Graphs defined on groups

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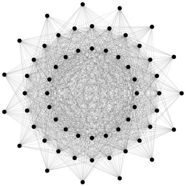

Lecture 7: Onward and upward
9 June 2021

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The obvious classes to take are the classes of nilpotent groups and soluble groups, giving the graphs $\mathrm{Nilp}(G)$ and $\mathrm{Sol}(G)$, lying above $\mathrm{Com}(G)$ in the hierarchy. If G is insoluble, then they both lie below $\mathrm{NGen}(G)$.

Minimal excluded groups

For our study, we need the analogue of the Miller–Moreno theorem:

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I do not know a complete classification of minimal insoluble groups. But if G is such a group, and S is its soluble radical, then G/S is a minimal (non-abelian) simple group; such groups were classified by Thompson (in his N-group paper), and all are 2-generated (without using CFSG). If we take generators of G/S and lift to G, the resulting elements generate G (by minimality, since the subgroup they generate is insoluble).

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6. *If G is insoluble, then* $E(\mathrm{Sol}(G)) \subseteq E(\mathrm{NGen}(G))$; *equality holds if and only if G is a minimal insoluble group.*

We have observed the first part already, while parts 5 and 6 follow from the fact that these groups are 2-generated.

Proof of 2 Suppose that $E(\mathrm{Com}(G)) = E(\mathrm{Nilp}(G))$. Then two elements from the same Sylow subgroup of G generate a nilpotent group; hence they commute. Conversely, if the Sylow subgroups are abelian, then a nilpotent subgroup is the product of its Sylow subgroups and hence is abelian.

Proof of 3 Suppose that $E(\mathrm{Nilp}(G)) = E(\mathrm{Sol}(G))$. If G is not nilpotent, it contains a minimal non-nilpotent subgroup, a Schmidt group, which is 2-generated and soluble, hence nilpotent, a contradiction. Conversely, if G is nilpotent, then $\mathrm{Nilp}(G)$ is complete.

Proof of 4 If $\mathrm{Com}(G)$ and $\mathrm{Sol}(G)$ coincide, then G is nilpotent with abelian Sylow subgroups, hence is abelian. The converse is clear.

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Now we are all set up for an analysis of these graphs along the lines we have seen for the lower terms in the hierarchy. But not much has been done on this, except for universality.

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Recall that any graph can be represented as an intersection graph of a linear hypergraph (two sets corresponding to adjacent vertices agreeing in one point). Now take the complement of Γ, represent it in this way, add dummy points so that each set has the same prime cardinality $p > 2$, and replace each set by a cycle with the given set as support.

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Two disjoint cycles commute, while two intersecting cycles generate the alternating group A_{2p-1}, which is not soluble.
Hence we have shown that Γ can be embedded in $\mathrm{Com}(G)$, $\mathrm{Nilp}(G)$, and $\mathrm{Sol}(G)$, for some group G (the group generated by all the p-cycles, which is a product of alternating groups).

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Indeed, for $p > 3$, we can catch another bird.
Assume that the complement of Γ is connected. (This can be achieved by adding an isolated vertex to Γ if necessary.) Then the group generated by the cycles is the alternating group on the union of the supports of the cycles.
Its Schur multiplier has order 2, so the lift of $C_p \times C_p$ to the Schur cover splits, and so disjoint p-cycles are joined in the deep commuting graph. So we can add $\mathrm{DCom}(G)$ to our tally.

The Engel graph

We define, for each positive integer k, and all $x, y \in G$, the element $[x, {}_k y]$ of G to be the left-normed commutator of x and k copies of y; more formally,

- $[x, {}_1 y] = [x, y] = x^{-1}y^{-1}xy$,
- for $k > 1$, $[x, {}_k y] = [[x, {}_{k-1} y], y]$.

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Abdollahi defined x and y to be adjacent if $[x, _k y] \neq 1$ and $[y, _k x] \neq 1$ for all k. To fit with the earlier philosophy I will redefine it to be the complement of this graph. If we do this then we have a similar situation to that arising with the power graph.

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We can define the directed Engel graph to have an arc from x to y if $[y,_k x] = 1$ for some k. Then the Engel graph is the graph in which x and y are joined if there is an arc from one to the other. The directed graph may also have a role to play here.

Zorn showed that, if a finite group G satisfies an Engel identity $[x,{}_k y] = 1$ for all x, y (for some k), then G is nilpotent; so the finite groups for which the directed Engel graph is complete are the same as those for which the nilpotency graph is complete. (For infinite groups, this is not true, though the result has been shown in a number of special cases.)

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So there is a close connection between the Engel graph and the nilpotency graph. But they are not equal in general. For example, in the group S_3, there is an arc of the directed Engel graph from each element of order 3 to each element of order 2, but not in the reverse direction.

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Question

What can be said about the relation between the Engel and nilpotency graphs? In particular, in which groups are they equal?

The Engel centre

Question

Which elements of the group G are joined to all others in the Engel graph?

I think the answer should be the Fitting subgroup, $F(G)$, the largest normal nilpotent subgroup of G. It is true that in the directed Engel graph, if $x \in F(G)$, then $x \to y$ for all $y \in G$. For $[y, x] \in F(G)$, and so repeated commutation with x results in the identity.

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But I cannot at present prove the converse.

More graphs

A wide generalisation has been considered by Lucchini and Nemmi. Let $\mathfrak{F}$ be a saturated formation of groups. (A formation is a class of groups closed under quotients and subdirect products; the formation $\mathfrak{F}$ is saturated if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$, where $\Phi(G)$ is the Frattini subgroup of G. Now the $\mathcal{F}$-graph of G can be defined by joining x and y if $\langle x, y \rangle \in \mathfrak{F}$.

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After removing the identity (which is fixed by all automorphisms), the graph is a disjoint union of cliques corresponding to the cyclic subgroups: 15 isolated points, 10 cliques of size 2 and 6 of size 4. So we have a normal subgroup n fixing all these, with structure $S_2^{10} \times S_4^6$, and the quotient is $S_{15} \times S_{10} \times S_6$; the product of the orders of these groups is the number quoted earlier.

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If we remove the identity, and then do closed twin reduction, and then open twin reduction, we reach a twin-free graph, the cokernel of the reduced power graph. It has 1210 vertices, and its automorphism group is exactly M_{11}. In fact this graph is bipartite, and the group acts with four orbits, of sizes 165 (twice), 220 and 660. Lurking in there is a very interesting bipartite graph with blocks of sizes 165 and 220, having diameter and girth equal to 10 (and again automorphism group M_{11}.

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It should be said that things are not always so interesting. It often happens that the original group "gets lost in the noise".

General results

To mention a couple of general results that we have seen implicitly:

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For each graph type χ in the hierarchy, and any non-trivial group G, the group $\mathrm{Aut}(\chi(G))$ has a non-trivial (usually large) normal subgroup which is a direct product of symmetric groups on the twin classes.

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The automorphism group of a cograph is built from the trivial group by the operations of direct product and wreath product with a symmetric group.

So, if $X(G)$ is a cograph, then G will almost certainly be "lost in the noise".

A question

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For which graph types X, and for which groups G, is it true that the automorphism group of the cokernel of $\mathrm{X}(G)$ is equal to the automorphism group of G?

As noted, this is the case for the power graph of M_{11}.

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I will sketch part of the proof, since is is a nice mixture of group theory and graph theory.

Proof.

The proof consists of showing that the hypothesis implies that $Z(G)$ has finite index in G. Now two elements in the same coset of $Z(G)$ commute, so an independent set cannot be larger than $|G : Z(G)|$.

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Every conjugacy class in G is finite.

For if not, let g lie in an infinite conjugacy class, and let S be an infinite set such that the elements $s^{-1}gs$ are all distinct. By Ramsey's Theorem, this set contains an infinite clique U. But if $u, v \in U$, then

$$[gu, gv] = u^{-1}g^{-1}v^{-1}g^{-1}guxg = (u^{-1}gu)^{-1}(v^{-1}gv) \neq 1,$$

since u and v commute. But then xU is an infinite independent set, a contradiction. $\square$

Other graphs

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Consider first the group C_{p^∞}, which can be defined either as the group of p-power roots of unity in $\mathbb{C}$, or as the group of rationals with p-power denominator in $\mathbb{Q}$ modulo $\mathbb{Z}$. This group has the property that its subgroups are finite cyclic groups of p-power order, one for each power of p. So the power graph is complete.

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The directed power graph does determine the prime, since the set of elements immediately above the identity in the preorder has cardinality $p - 1$.

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Nevertheless, something can be proved:

Theorem

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So G is locally finite, a result of Shitov.

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Recall that $Z_{\mathrm{EPow}}(G)$ is a subgroup of G, called the cyclicizer. It is the set of elements $x \in G$ such that, for all $y \in G$, $\langle x, y \rangle$ is cyclic.

Cliques and colourings

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Of course there is no such result for the commuting graph, since there are arbitrarily large abelian groups. We have the following result for the case where the numbers are finite.

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Proof.

The power graph of an infinite cyclic group $\langle g \rangle$ contains an infinite clique $\{g^{2^n} : n \geq 0\}$. So a group satisfying any of the first four conditions is a torsion group. Now the results are proved just as for finite groups. $\square$

Directing the power graph

We saw that the power graph determines the directed power graph up to isomorphism in the case of a finite group. This fails for infinite groups: the groups C_{p^∞}, for primes p, all have power graph which is countable and complete, but their directed power graphs are all different.

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We saw that the power graph determines the directed power graph up to isomorphism in the case of a finite group. This fails for infinite groups: the groups C_{p^∞}, for primes p, all have power graph which is countable and complete, but their directed power graphs are all different.
But the result does hold for torsion-free groups. Indeed, a theorem of Zahirović shows clearly the important role played by C_{p^∞}:

Theorem

Let G and H be infinite groups with $\mathrm{Pow}(G) \cong \mathrm{Pow}(H)$. Suppose that G has no subgroup $K \cong C_{p^\infty}$ with the property that, for any cyclic subgroup L of G, either $L \leq K$ or $L \cap K = \{1\}$. Then $\mathrm{DPow}(G) \cong \mathrm{DPow}(H)$.

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In some of these, the definitions of the commuting graph, power graph, and so on, can be adapted almost without change, but the theory may be quite different.

In other cases, like rings, there are new opportunities for defining graphs which provide information about the structure: we will see such graphs as the zero-divisor graph and unit graph of a ring.