Graphs defined on groups

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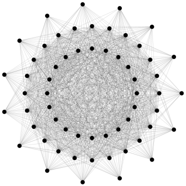

Lecture 8: Other worlds
9 June 2021

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The definition of the commuting graph of an arbitrary magma is straightforward. We cannot expect to define, say, the deep commuting graph. But what about the power graph?

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But different definitions (for example, $x^{n+1} = x \circ (x^n)$) could give different results. We define a magma to be power-associative if the value of x^n is independent of the definition. This can be expressed by the equations

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Now in any magma we could define the directed power graph by the rule that $a \to b$ if $b = a^n$ for some $n \in \mathbb{N}$, and the power graph by the rule that $x \sim y$ if either $x \to y$ or $y \to x$. But this is not likely to make much sense unless the magma is power-associative.

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The proof is immediate.
Now one of our early theorems about groups asserted that if two groups have isomorphic power graphs, then they have isomorphic directed power graphs.

Question

What assumptions on a magma are required for this theorem to hold?

Quasigroups and loops

The Cayley table of a magma M of order n is the $n \times n$ array with rows and columns indexed by M, having (x, y) entry $x \circ y$. (Note that some people reserve the term "Cayley table" for groups, and would call this an "operation table".)

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A magma is a quasigroup if it satisfies the left and right division laws; that is, for any a and b, each of the equations $x \circ a = b$ and $a \circ y = b$ has a unique solution. A quasigroup is a loop if it has an identity element.

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In terms of the Cayley table, a quasigroup is a magma for which each element occurs once in each row and once in each column of the Cayley table (in other words, the Cayley table is a Latin square). If the quasigroup is a loop, and we order it so that the identity is the first element, then the first row agrees with the row of column labels, and the first column agrees with the column of row labels.

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- *$x(xy) = (xx)y$, $y(xx) = (yx)x$, and $(xy)x = x(yx)$;*
- *any 2-generated subloop is associative.*

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Here is a sketch of the proof. It follows the corresponding proof for groups. If only the identity is a dominating vertex, the same argument works.

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The proof shows that, if the order of M is not a 2-power, then it must be a cyclic group; if it is a 2-power, then M is cyclic, generalised quaternion, or generalised octonion, depending on whether the largest complete subgraph has size $|M|$, $|M|/2$ or $|M|/4$.

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Question

Can anything similar be done for other classes of loops?

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As far as I am aware, the enhanced power graph of a power-associative loop has not been studied, although the definition presents no problems.
It is probably true that the deep commuting graph makes little sense outside the context of groups.

Semigroups and monoids

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An inverse semigroup is a semigroup in which, for each element x, there is a unique generalised inverse y satisfying $xyx = x$ and $yxy = y$. The element y is denoted by x^*.

Random walks

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So inverse semigroups might be candidates for the above question ...

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In what follows, “ring” will mean “commutative ring with identity”.

I will talk about the zero-divisor graph, though other graphs such as the unit graph have been considered.

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This uses some standard results from ring theory. The radical I of a finite ring R is nilpotent, and hence R is complete in the I-adic topology.

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This graph was introduced by Anderson and Livingston in 1999.

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Proof.

We use Boolean rings: the elements are all subsets of a set X, with symmetric difference for addition and intersection for multiplication. Now $ab = 0$ if and only if a and b are disjoint. So if we represent the given graph as an intersection graph, it is naturally embedded in the zero-divisor graph of a Boolean ring. $\square$

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I am grateful to G. Arun Kumar for this proof.

Local rings

In a finite ring, every non-zero element is either a zero-divisor or invertible. (If multiplication by a is not injective, then a is a zero-divisor; if it is surjective, then a is a unit.) So in a local ring, the zero-divisors are the non-zero elements of the maximal ideal.

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However, not all finite local rings have their maximal ideals principal, and the zero-divisor graph is not always a threshold graph. (I am grateful to T. Kavaskar for an example.) The question above remains unanswered.

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