Generalisations of EPPO groups using graphs

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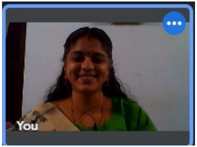
This project is one of several spin-offs from a research discussion on Graphs and Groups run from Kochi, India, over the summer.



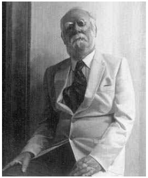
The organisers were Vijayakumar Ambat and Aparna Lakshmanan at CUSAT, to whom I am grateful.

EPPO groups

An EPPO group is a finite group in which every element has prime power order.

- A group of prime power order is an EPPO group.
- Of the two groups of order 6, the dihedral group is an EPPO group (all elements have orders 1, 2 or 3) but the cyclic group is not.
- Thinking about this example, we see that a nilpotent group (which is the direct product of its Sylow subgroups) is an EPPO group if and only if it has prime power order.

History

EPPO groups were introduced by Graham Higman in the 1950s; he classified the soluble ones.
In the 1960s, as a spin-off from the discovery of his infinite family of simple groups, Michio Suzuki classified the simple EPPO groups.
All EPPO groups have been classified. It seems that the first person to do this was Rolf Brandl in 1981.

Mystery

My identification of Brandl and the facts below are not 100% certain; I haven't seen Brandl's paper, only the review in *Zentralblatt*. It is published in *Bolletin Unione Matematica Italiana* (5) **18** A (1981), 491–493. This journal went through many series, and had several separate sections, and I have been unable to locate this paper on the internet.

Since then, a number of authors have rediscovered the result. Most surprisingly, Hermann Heineken in 2008. At the end of his paper, which also discusses locally finite EPPO groups, Heineken has a note added in proof, "The author was not aware of this paper", and gives the reference to Brandl.

According to the Mathematics Genealogy website, Brandl was a student of Heineken, getting his PhD in 1979, two years before his paper on EPPO groups.

Earlier this year, I needed the list of EPPO groups, so I asked Natalia Maslova if she had one. She sat down and worked it out herself.

Theorem

An EPPO group G satisfies one of the following:

- $|\pi(G)| = 1$ *and G is a p-group.*
- $|\pi(G)| = 2$ *and G is a soluble Frobenius or 2-Frobenius group (see later).*
- $|\pi(G)| = 3$ *and* $G \in \{A_6, \mathrm{PSL}_2(7), \mathrm{PSL}_2(17), M_{10}\}$.
- $|\pi(G)| = 3$, $G/O_2(G)$ *is* $\mathrm{PSL}_2(2^n)$ *for* $n \in \{2, 3\}$ *and if* $O_2(G) \neq \{1\}$, *then* $O_2(G)$ *is the direct product of minimal normal subgroups of G, each of which is of order* 2^{2n} *and as* $G/O_2(G)$*-module is isomorphic to the natural* $\mathrm{GF}(2^n)\,\mathrm{SL}_2(2^n)$*-module.*
- $|\pi(G)| = 4$ *and* $G \cong \mathrm{PSL}_3(4)$.
- $|\pi(G)| = 4$, $G/O_2(G)$ *is* $\mathrm{Sz}(2^n)$ *for* $n \in \{3, 5\}$, *and if* $O_2(G) \neq \{1\}$, *then* $O_2(G)$ *is the direct product of minimal normal subgroups of G, each of which is of order* 2^{4n} *and as* $G/O_2(G)$*-module is isomorphic to the natural* $\mathrm{GF}(2^n)\,\mathrm{Sz}(2^n)$*-module of dimension* 4.

Gruenberg and Kegel

In the 1960s, while investigating integral representations of finite groups, Karl Gruenberg and Otto Kegel defined the prime graph of a finite group, now more usually referred to as the Gruenberg–Kegel graph or GK graph.

The Gruenberg–Kegel graph

The vertex set of the GK graph of a group G is the set of prime divisors of $|G|$. (Equivalently, by Cauchy's Theorem, the set of prime orders of elements of G.) Two vertices p and q are joined if G contains an element of order pq. This tiny graph carries a lot of information about the group.

- A glance at the 𝔸𝕋𝕃𝔸𝕊 of finite groups shows, for example, that the Mathieu group M_{11} has vertex set $\{2, 3, 5, 11\}$ and just a single edge $\{2, 3\}$.
- G is an EPPO group if and only if its GK graph is a null graph (that is, has no edges).

Frobenius and 2-Frobenius groups

The group G is a Frobenius group if it has a proper subgroup H (called a Frobenius complement) with the property that $H \cap H^g = \{1\}$ for all $g \in G \setminus H$. The symmetric group S_3 is an example.

Frobenius showed that, if N is the set of elements lying in no conjugate of H, together with the identity, then N is a normal subgroup of G, called the Frobenius kernel. Moreover, Thompson showed that the Frobenius kernel is nilpotent, and Zassenhaus determined the structures of Frobenius complements.

The group G is a 2-Frobenius group if it has a chain of normal subgroups $\{1\} \lhd N \lhd M \lhd G$ such that

- M is a Frobenius group with Frobenius kernel N;
- G/N is a Frobenius group with Frobenius kernel M/N.

The symmetric group S_4 is an example.

The theorem of Gruenberg and Kegel

The main theorem of Gruenberg and Kegel was a structure theorem for groups whose GK graph is disconnected. It was contained in an unpublished manuscript, and published by J. S. Williams (a student of Gruenberg) in 1981.

Theorem

Let G be a finite group whose GK graph is disconnected. Then one of the following holds:

- *G is a Frobenius or 2-Frobenius group;*
- *G is an extension of a nilpotent π-group by a simple group by a π-group, where π is the set of primes in the connected component containing 2.*

Which simple groups can occur in the second conclusion of the theorem? This was investigated by Williams, though he was unable to deal with groups of Lie type in characteristic 2. The work was completed by Kondrat'ev in 1989, and some errors corrected by Kondrat'ev and Mazurov in 2000.

GK graph and EPPO groups

We have seen that G is an EPPO group if and only if its Gruenberg–Kegel graph has no edges.
So we could regard groups whose GK graph has only a few edges (in some sense) as being a generalisation of EPPO groups. If the graph is disconnected, then we have good structural information about the group (though less than a complete classification).
I turn now to some other ways to generalise the EPPO groups using graphs.

Other graphs

For some different generalisations, I define two more graphs. In both cases, the vertex set of the graph is the group G.

- g and h are joined in the power graph $\mathrm{Pow}(G)$ of G if one of them is a power of the other.
- g and h are joined in the enhanced power graph $\mathrm{EPow}(G)$ of G if they are both powers of an element k (in other words, if the group $\langle g, h \rangle$ they generate is cyclic).

We see that the edge set of $\mathrm{Pow}(G)$ is contained in that of $\mathrm{EPow}(G)$. (In graph theory language, $\mathrm{Pow}(G)$ is a spanning subgraph of $\mathrm{EPow}(G)$.)

EPPO groups reappear

Theorem

The finite group G satisfies $\mathrm{Pow}(G) = \mathrm{EPow}(G)$ *if and only if G is an EPPO group.*

Proof.

If G fails to be an EPPO group, then it contains an element g of order pq for some primes p and q. Then g^p and g^q are joined in the enhanced power graph (since both are powers of g) but not in the power graph.

Conversely, if G is an EPPO group, and $\langle g, h \rangle$ is cyclic, then it has prime power order, and so one of g and h generates this group, say g; then h is a power of g. □

Thus the classification of EPPO groups gives us the groups G for which $\mathrm{Pow}(G) = \mathrm{EPow}(G)$.

Generalisations

Let p be any graph parameter which is monotonic: that is, adding edges to a graph cannot decrease the value of p.

Then $p(\mathrm{Pow}(G)) \leq p(\mathrm{EPow}(G))$. Asking for which groups equality holds is a generalisation of asking for which groups $\mathrm{Pow}(G) = \mathrm{EPow}(G)$, that is, the EPPO groups.

Here is a fairly easy example. The clique number of a graph is the size of the largest complete subgraph.

Theorem

For a finite group G, the power graph and enhanced power graph have equal clique number if and only if the largest order of an element of G is a prime power.

Clearly this class of groups includes the EPPO groups!

Proof

We use the fact that if S is a subset of a finite group, and any two elements of S generate a cyclic group, then $\langle S \rangle$ is cyclic. (This is an exercise for the reader.)

Thus any clique in the enhanced power graph is contained in a cyclic subgroup of G, and the largest clique is the largest cyclic subgroup.

Now a cyclic group is a clique in the power graph if and only if it has prime power order. (An easier exercise for the reader.)

So the largest clique in the power graph is as big as the largest clique in the enhanced power graph if and only if the largest cyclic subgroup of G has prime power order.

An example

For which prime powers q do the power graph and enhanced power graph of $\mathrm{PGL}(2, q)$ have the same clique number?
The maximal order of an element in this group is $q + 1$, so the necessary and sufficient condition is that $q + 1$ is also a prime power.
According to the Catalan conjecture, this occurs only in one of the following cases:

- q is a power of 2 and $q + 1$ is a Fermat prime;
- q is a Mersenne prime and $q + 1$ is a power of 2;
- $q = 8$, $q + 1 = 9$.

(The Catalan conjecture asserts that the only solution of $x^a - y^b = 1$ in positive integers x, y, a, b with $a, b > 1$ is $3^2 - 2^3 = 1$. It was proved by Mihăilescu in 2002.)

Matching number

I will illustrate with one somewhat striking example. The matching number $\mu(\Gamma)$ of a graph Γ is the maximum number of pairwise disjoint edges in Γ. This is clearly a monotonic graph parameter.

With V. V. Swathi and M. S. Sunitha from Calicut, I proved:

Theorem

For any finite group G, the matching numbers of $\mathrm{Pow}(G)$ *and* $\mathrm{EPow}(G)$ *are equal.*

The small surprise is that we cannot calculate the matching number of $\mathrm{Pow}(G)$ for all groups G, merely give upper and lower bounds.

The strategy of the proof is to show that, given a matching in the enhanced power graph, we can replace its edges by edges of the power graph to find another matching of the same size.

Cographs

To describe the other generalisation, we have to make a detour. A cograph is a graph which doesn't contain the 4-vertex path as an induced subgraph. These are also referred to as complement-reducible graphs, hereditary Dacey graphs, or (my favourite) N-free graphs. The variety of names indicate the importance of this class.
Since the path P_4 is isomorphic to its complement, the class of cographs is self-complementary. In fact, it is the smallest class of graphs containing the 1-vertex graph and closed under disjoint union and complementation. This means that the class has very nice algorithmic properties, which don't concern us here.

The power graph of a p-group is a cograph

Recall that in the power graph, g and h are joined if one is a power of the other. So the graph is naturally a directed graph, with an arc $g \to h$ if h is a power of g. It is easily seen that this relation is transitive.
Hence, if we have an induced P_4 in a cograph, directions must alternate:

$$a \to b \leftarrow c \to d.$$

Now in a p-group, if $c \to b$ and $c \to d$, then b and d lie in a cyclic group of prime power order, so one is a power of the other. Hence there can be no induced P_4:

Theorem

The power graph of a group of prime power order is a cograph.

The power graph of an EPPO group is a cograph

This follows easily from the previous result. Hence the following problem is a generalisation of the problem of determining EPPO groups:

Problem

Determine the finite groups whose power graph is a cograph.

I have worked on this problem with Pallabi Manna and Ranjit Mehatari from Rourkela. Our first theorem states:

Theorem

If G is a nilpotent group, then the power graph of G is a cograph if and only if either G has prime power order, or $G = C_{pq}$ where p and q are primes.

Recall that a nilpotent EPPO group has prime power order. The addition of the groups C_{pq} has a big effect on the class of groups!

Simple groups whose power graph is a cograph

Using this result, it is possible to show the following.

Theorem

Let G be a finite simple group whose power graph is a cograph. Then one of the following holds:

- $G = \mathrm{PSL}(2, q)$ *for a prime power q, where each of* $(q+1)/\gcd(q+1, 2)$ *and* $(q-1)/\gcd(q-1, 2)$ *is either a prime power or the product of two primes;*
- $G = \mathrm{Sz}(q)$ *for q an odd power of* 2, *where each of* $q-1$, $q+\sqrt{2q}+1$ *and* $q-\sqrt{2q}+1$ *is either a prime power or the product of two primes;*
- $G = \mathrm{PSL}(3, 4)$.

Note that $\mathrm{PSL}(2, 11)$ and M_{11} have identical GK graphs, but the power graph of the first is a cograph, that of the second is not.

A problem for number theorists

Problem

Are there infinitely many values of q for which $\mathrm{Pow}(\mathrm{PSL}(2,q))$ *is a cograph?*

For example, the values of d up to 200 for which the power graph of $\mathrm{PSL}(2,2^d)$ is a cograph are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.
Similar (possibly easier) question for $\mathrm{Sz}(q)$.

And finally ...

The power graph of a finite group is the comparability graph of a partial order (this means two elements are joined if and only if they are comparable in the partial order).
For the relation $x \to y$ is reflexive and transitive (that is, a partial preorder. In a partial preorder, the relation $x \equiv y$ if $x \to y$ and $y \to x$ is an equivalence relation, whose equivalence classes are partially ordered by $\to$. Now simply put any total order on each equivalence class to get the required partial order.
Hence the power graph is perfect (this means that every induced subgraph has clique number equal to chromatic number), by Dilworth's Theorem.

So one further generalisation of the classification of EPPO groups is:

Problem

- *For which finite groups is the enhanced power graph the comparability graph of a partial order?*
- *For which finite groups is the enhanced power graph perfect?*

... for your attention.