The Gruenberg–Kegel graph and graphs defined on groups

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Plan

The Gruenberg–Kegel graph of a finite group G has as vertices the set of prime divisors of the order of G, with an edge from p to q if and only if G contains an element of order pq. It was introduced for the study of the group ring of G.

There are a number of graphs which have been studied, whose vertex set is G, going back to the commuting graph introduced by Brauer and Fowler in their classic paper of 1955. Others include the power graph, and more recently the enhanced power graph and the deep commuting graph.

It turns out that the (very small) Gruenberg–Kegel graph carries a lot of information about the (much larger) graphs on the group G; this is the topic I will be exploring.

Outline

After introducing the four graphs to be considered, I will discuss four topics:

▶ If two groups G and H have isomorphic commuting graphs (or indeed any of the other types), then they have the same GK graph.

▶ If $Z(G) = 1$, then the reduced commuting graph of G (the induced subgraph of the commuting graph on the non-identity elements of G) is connected if and only if the GK graph is connected.

▶ The enhanced power graph is equal to the power graph if and only if the GK graph has no edges. Such groups can be determined.

▶ There is a necessary condition, and a sufficient condition, in terms of the GK graph for the power graph of G to be a cograph; but there is no necessary and sufficient condition in these terms. Moreover, even for the groups $\mathrm{PSL}(2, q)$, this leads to difficult number-theoretic questions.

The Gruenberg–Kegel graph

I knew Karl Gruenberg well. He was my colleague at Queen
Mary, University of London, from the time I moved there in
1986 until his death in 2007. His main work was in the
cohomology and integral representation of groups. It is a
pleasure for me to remember him in this talk.
I was less well acquainted with Otto Kegel, but he visited
Oxford once a week for a term when I was a student there to
lecture on locally finite groups. I guess this was when he was
visiting Bert Wehrfritz and they were writing a book on them.

The Gruenberg–Kegel graph, sometimes called the prime graph, of a finite group G was introduced by Gruenberg and Kegel in an unpublished manuscript in 1975. They were concerned with the decomposability of the augmentation ideal of the integral group ring of G.

The vertex set of the graph is the set of prime divisors of the order of G (equivalently, by Cauchy's theorem, the set of orders of elements of prime order in G). It has an edge joining p and q if and only if G contains an element of order pq (equivalently, there are commuting elements of orders p and q).

Their main theorem was a characterisation of groups whose GK graph is disconnected. The result was subsequently published by Williams (a student of Gruenberg) in 1981.

The Gruenberg–Kegel graph has received a lot of attention since Williams' paper. Natalia Maslova and others in this session will tell you more about its properties.

The number of prime divisors of $|G|$ is very much smaller than the order of G. I will be concerned with several graphs defined to have vertex set G and various joining rules reflecting the algebraic structure of G. So these graphs are much larger than the GK graph. Nevertheless, a surprising amount of their structure is determined by the GK graph.

This is the subject of my talk. I will introduce four particular graphs that I will speak about, and present results connecting the GK graph of a group G with one of these four defined on G.

Dramatis Personae, 1: the commuting graph

The commuting graph was the first of the four to be defined; it appeared in the famous paper of Brauer and Fowler on centralisers of involutions in finite groups in 1955.

The commuting graph of G has vertex set G; vertices g and h are joined if and only if $gh = hg$. (This definition would put a loop at every vertex; we silently suppress these.)

Here are the commuting graphs of the two non-abelian groups of order 8: $D_8 = \langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$.
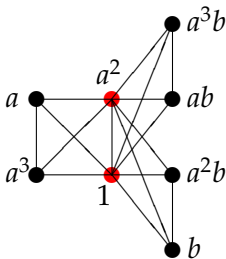
Dramatis Personae, 2: the power graph

Next on the scene, much later, was the power graph, defined by Kelarev and Quinn in 1999.

The directed power graph of G has vertex set G; there is an arc from g to h if h is a power of g. Again we ignore loops. The power graph of G is obtained by ignoring the directions on the arcs (and suppressing one of them if each of g and h is a power of the other); in other words, g and h are joined if one is a power of the other.

The power graph does not uniquely determine the directions; but it is known that if two groups have isomorphic power graphs, then they have isomorphic directed power graphs.

The graph shown earlier is the power graph of Q_8, but not of D_8.

Dramatis Personae, 3: the enhanced power graph

The enhanced power graph is more recent. The complementary graph was defined by Abdollahi and Hassanabadi in 2007 under the name "noncyclic graph"; I will use the description appearing in a paper of Aalipour *et al.* in 2017.

The enhanced power graph has vertex set G; we join g to h if and only if there is an element k such that both g and h are powers of k. Equivalently, g and h are joined if and only if $\langle g, h \rangle$ is cyclic.

We saw that the power graph determines (up to isomorphism) the directed power graph, and hence the enhanced power graph. The converse is also true; if two graphs have isomorphic enhanced power graphs, then they have isomorphic power graphs.

Dramatis Personae, 4: the deep commuting graph

The final actor, the deep commuting graph, is not yet published, but can be found in a paper by Bojan Kuzma and me on the arXiv.

The deep commuting graph has vertex set G; vertices g and h are joined if and only if their inverse images in every central extension of G commute. (That is, if $Z \leqslant Z(H)$ with $H/Z \cong G$, and aZ and bZ are the cosets of Z corresponding to g and h, then we require that a and b commute (in every such extension).

We showed that it suffices to take a single central extension of G, namely a Schur cover (where $Z \leqslant Z(H) \cap H'$ and subject to this Z is as large as possible). It is independent of the choice of Schur cover.

For example, the Klein group of order 4 has two Schur covers, the dihedral and quaternion groups of order 8; so the deep commuting graph of V_4 is the star $K_{1,3}$.

The hierarchy

The four graphs defined in this way on G form a hierarchy, in the order power graph, enhanced power graph, deep commuting graph, commuting graph, in the sense that the edge set of each is contained in the edge set of the next.

This is clear in all cases except the enhanced power graph and deep commuting graph. So suppose that $\{g, h\}$ is an edge of the enhanced power graph, so that $\langle g, h \rangle = \langle k \rangle$ for some k. Now let H be a central extension of G with kernel Z. Let a, b, c be representatives of the cosets of Z in H corresponding to g, h, k. Then $\langle Z, a, b \rangle = \langle Z, c \rangle$, which (as a cyclic extension of a central subgroup) is abelian; so a and b commute.

One further observation: for each type Γ in the hierarchy, $\Gamma(G)$ is invariant under the automorphism group of G.

Graphs in the hierarchy determine the GK graph

Theorem
Let $\Gamma(G)$ denote one of the four types of graph in the hierarchy. If G and H are groups with $\Gamma(G) \cong \Gamma(H)$, then the Gruenberg–Kegel graphs of G and H are equal.

Proof.
Consider first the enhanced power graph or the commuting graph. A maximal clique in one of these graphs is a maximal cyclic (resp. abelian) subgroup of G. So p and q are joined in the GK graph if and only if there is a maximal clique of the graph having order divisible by pq.

A similar but slightly more elaborate proof works for the deep commuting graph.

Finally, we saw that if the power graphs of G and H are isomorphic, then so are the enhanced power graphs. $\qquad \square$

Connectedness

One of the most studied questions about the commuting graph, going back to the Brauer–Fowler paper, is connectedness.
Of course the commuting graph of G is connected, since any element of the centre is joined to all other elements of the group. So it makes sense to define the reduced commuting graph of G to be the induced subgraph of the commuting graph on $G \setminus Z(G)$.
Much is known about this, which I will not describe here. I want just to mention one result which fits my theme.

Theorem
Let G be a finite group with $Z(G) = 1$. Then the reduced commuting graph of G is connected if and only if the Gruenberg–Kegel graph of G is connected.

Equality

Since our four graphs on G form a hierarchy, a natural question is:

Question

For which groups G do a pair of graphs from the hierarchy on G coincide?

At least partial results on all of these $\binom{4}{2} = 6$ questions are known. My interest here is in graphs for which the power graph and enhanced power graph coincide.

EPPO groups

An EPPO group is a group in which every element has prime power order. There has been a lot of work on these, and it is a pleasure to mention some of my former colleagues in Oxford: Graham Higman, Patrick Martineau and Brian Stewart.

Theorem

For a finite group G, the following three conditions are equivalent:

- ▶ *G is an EPPO group;*
- ▶ *the Gruenberg–Kegel graph of G has no edges;*
- ▶ *the power graph of G is equal to the enhanced power graph.*

The proof

Equivalence of the first two conditions is immediate from the definition.

If two elements have prime power order and generate a cyclic group, then one of them is a power of the other. So in an EPPO group, the power graph and enhanced power graph are equal. Conversely, if G is not an EPPO group, then it has an element which is not of prime power order; some power of this element has order a power of two primes, and so gives an edge in the GK graph. $\qquad\square$

On the next slide, $\pi(G)$ is the set of prime divisors of $|G|$. The theorem (in my paper with Natalia Maslova) is the culmination of work on this problem.

The classification

Theorem

An EPPO group G satisfies one of the following:

- $|\pi(G)| = 1$ *and G is a p-group.*

- $|\pi(G)| = 2$ *and G is a solvable Frobenius or 2-Frobenius group.*

- $|\pi(G)| = 3$ *and* $G \in \{A_6, \mathrm{PSL}_2(7), \mathrm{PSL}_2(17), M_{10}\}$.

- $|\pi(G)| = 3$, $G/O_2(G)$ *is* $\mathrm{PSL}_2(2^n)$ *for* $n \in \{2, 3\}$ *and if* $O_2(G) \neq \{1\}$, *then* $O_2(G)$ *is the direct product of minimal normal subgroups of G, each of which is of order* 2^{2n} *and as a* $G/O_2(G)$-*module is isomorphic to the natural* $\mathrm{GF}(2^n)SL(2^n)$-*module.*

- $|\pi(G)| = 4$ *and* $G \cong \mathrm{PSL}_3(4)$.

- $|\pi(G)| = 4$, $G/O_2(G)$ *is* $\mathrm{Sz}(2^n)$ *for* $n \in \{3, 5\}$, *and if* $O_2(G) \neq \{1\}$, *then* $O_2(G)$ *is the direct product of minimal normal subgroups of G, each of which is of order* 2^{4n} *and as a* $G/O_2(G)$-*module is isomorphic to the natural* $\mathrm{GF}(2^n)\,\mathrm{Sz}(2^n)$-*module of dimension* 4.

Cographs

A finite graph Γ is a cograph if it has no induced subgraph isomorphic to the 4-vertex path. Since this graph is self-complementary, the complement of a cograph is a cograph. Here are a few properties of cographs:

Theorem

▶ *If a cograph on more than one vertex is connected, then its complement is disconnected.*

▶ *Cographs form the smallest non-empty class of graphs which is closed under the operations of complement and disjoint union; so every cograph can be built from the 1-vertex graph by these operations.*

▶ *The automorphism group of a cograph can be built from the trivial group by the operations "direct product" and "wreath product with a symmetric group".*

Twins and twin reduction

Two vertices in a finite graph Γ are twins if they have the same neighbours (possibly excepting one another).

Being twins is an equivalence relation; the automorphism group of the graph has a normal subgroup acting as the symmetric group on each twin class.

If we collapse a pair of twins to a single vertex, and continue this until there are no more pairs of twins, the result (which I will call the cokernel of the graph) is independent of the sequence of reductions.

Theorem
Γ *is a cograph if and only if its cokernel is the* 1-*vertex graph.*

Graphs in the hierarchy always have twins

Theorem
For each of the types X *of graph introduced earlier, if G is a non-trivial group, then* X(G) *has non-trivial twin relation.*

Proof.
If g is an element of G with order $m > 2$, and $\gcd(m, d) = 1$, then g and g^d are closed twins (i.e. have the same closed neighbourhood).

The only case remaining is that when G is an elementary abelian 2-group, in which case X(G) is either complete (for the commuting graph) or a star (for the others). $\qquad\square$

So if you are interested in the automorphism group, you might first want to carry out twin reduction. Of course, if the graph is a cograph, the result will be the trivial graph ...

When is the power graph a cograph?

For any of the four types of graph in our hierarchy, and any group G of order greater than 1, the corresponding graph on G has non-trivial twin relation. For, if g is an element of order $m \geqslant 2$, and d is coprime to m, then g and g^d are twins. In the excluded case, G is an elementary abelian 2-group; this is easily dealt with.

So we can apply twin reduction, and it is of some interest to know whether we reach the 1-vertex graph (in other words, whether the graph is a cograph).

In the remainder of the talk, I will consider this for the power graph. I will give a necessary condition, and a sufficient condition, for the power graph to be a cograph, in terms of the GK graph of G.

Power graph and GK graph

Theorem

▶ *Suppose that all connected components of the GK graph are singletons (that is, G is an EPPO group). Then the power graph of G is a cograph.*

▶ *Suppose that G is non-solvable, and that the power graph of G is a cograph. Then every connected component of the GK graph of G except possibly the component containing the prime* 2 *has size at most* 2.

There is no necessary and sufficient condition just in terms of the GK graph. For the groups PSL(2, 11) and M_{11} have the same GK-graph; the power graph of the first is a cograph, but the power graph of the second is not (its cokernel has 1212 vertices and automorphism group M_{11}).

What happens for $\mathrm{PSL}(2, q)$?

Theorem

For a prime power q, let l and m be $q - 1$ and $q + 1$ if q is even, or $(q - 1)/2$ and $(q + 1)/2$ if q is odd. Then the power graph of $G = \mathrm{PSL}(2, q)$ is a cograph if and only if each of l and m are either a prime power or a product of two distinct primes.

Question

Are there infinitely many prime powers q for which the power graph of $\mathrm{PSL}(2, q)$ is a cograph?

This is probably quite a hard number-theoretic problem. The values of d up to 200 for which the conditions of the theorem hold for $q = 2^d$ are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.

Other simple groups

The table on the next slide gives the sizes of cokernels of various graphs defined on the first few finite non-abelian simple groups G by order.

I have added in one further graph, the non-generating graph, in which g and h are joined if and only if $\langle g, h \rangle \neq G$. Since G is 2-generated, this graph is not complete; since G is not abelian, its edge set contains the edge set of the commuting graph of G. Recall that a graph is a cograph if and only if its cokernel has 1 vertex.

Question

For the various types Γ of graph, for which simple groups G is $\Gamma(G)$ a cograph?

G	$|G|$	Power	E-Power	D-Com	Com	N-Gen
A_5	60	1	1	1	1	32
PSL(2,7)	168	1	1	1	44	79
A_6	360	1	1	1	92	167
PSL(2,8)	504	1	1	1	1	128
PSL(2,11)	660	1	1	1	112	244
PSL(2,13)	1092	1	1	1	184	366
PSL(2,17)	2448	1	1	1	308	750
A_7	2520	352	352	352	352	842
PSL(2,19)	3420	1	1	1	344	914
PSL(2,16)	4080	1	1	1	1	784
PSL(3,3)	5616	756	756	808	808	1562
PSU(3,3)	6048	786	534	499	499	1346
PSL(2,23)	6072	1267	1	1	508	1313
PSL(2,25)	7800	1627	1	1	652	1757
M_{11}	7920	1212	1212	1212	1212	2444

The table suggests various conjectures, some of which can be proved. For example:

Theorem
The non-generating graph of a non-abelian finite simple group is not a cograph.

Proof.
The results of a number of authors show that the reduced non-generating graph (with the identity removed) and its complement are both connected, with small diameter. □

Question
Find, or estimate, the number of vertices in the cokernel of the non-generating graph of a finite simple group.

References

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for your attention. Stay well!