Graphs defined on groups

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Algebraic graph theory is the area where these two very different subjects can meet and have a productive relationship.

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I hope to show you a part of algebraic graph theory which is full of interesting unsolved problems, and give you a taste of some of these.

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The story begins some time later than Cayley, with a paper in 1955 by Brauer and Fowler. There are several remarkable things about this paper.

- As in all of Brauer's early papers, a group is denoted by $\mathfrak{G}$, its order by g, and a typical element by G.
- The main theorem of the paper is that, given a finite group H, there are only finitely many finite simple groups containing an involution whose centralizer is H. This could be regarded as the beginning of the Classification of Finite Simple Groups, in which characterizations of simple groups by the centralizer of an involution plays such a big part. But this result is not formally stated as a theorem in the paper.

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The identity (or indeed, elements of the centre) are joined to all other vertices; so distances in this graph are not very interesting. So it is fairly common to do as Brauer and Fowler implicitly did, and delete the vertices in $Z(G)$.

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The identity (or indeed, elements of the centre) are joined to all other vertices; so distances in this graph are not very interesting. So it is fairly common to do as Brauer and Fowler implicitly did, and delete the vertices in $Z(G)$.
I will not do so; I will explain why shortly. So for me the vertex set of the commuting graph is G.

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This is not the complete *dramatis personae*, just the big stars. Some others will come in later. Indeed you can imagine some for yourself. Noting that x and y are joined in the commuting graph if and only if $\langle x, y \rangle$ is abelian, we could define a graph where the joining rule is $\langle x, y \rangle$ is nilpotent, or solvable, or ...

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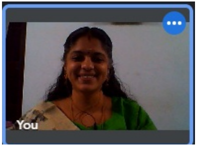

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In order to get started, we observe that these graphs form a hierarchy; each is contained in the next as a spanning subgraph. This is the main reason for taking the vertex set in each case to be the whole group.

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- The non-generating graph $\mathrm{NGen}(G)$: $\langle x, y \rangle \neq G$.
- The complete graph.

Each is contained in the next, except that the commuting graph is contained in the non-generating graph if and only if G is either non-abelian or has more than two generators.

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- The power graph is null if and only if G is the trivial group (for the identity is joined to all other vertices).
- The non-generating graph is complete if and only if G is not 2-generated.
- The commuting graph is equal to the non-generating graph if and only if G is a minimal non-abelian group. Such groups were determined by Miller and Moreno in 1904.

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The groups in each of these classes have been determined. Before explaining this, let me mention another graph associated with a finite group.

The Gruenberg–Kegel graph

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- *G is a Frobenius or 2-Frobenius group; or*
- *G is an extension of a nilpotent π-group by a simple group by a π-group, where π is the set of primes in the connected component containing 2.*

EPPO groups

The group G is an EPPO group ("Elements of Prime Power Order") if every element of G has prime power order. These groups were studied by Higman in the 1950s; he determined the solvable ones. Following the discovery of his infinite family of simple groups, Suzuki was able to determine the simple EPPO groups. Now we have a complete classification.

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In the other case, let $O(G)$ be the maximal normal subgroup of G of odd order. By the Brauer–Suzuki theorem, $H = G/O(G)$ contains a unique subgroup Z of order 2, and H/Z has dihedral Sylow 2-subgroups, so is determined by the Gorenstein–Walter theorem. Then a cohomogical argument due to Glauberman shows that if a group K has dihedral Sylow 2-subgroups, then there is a unique H (up to isomorphism) containing a unique subgroup Z of order 2 such that $H/Z \cong K$.

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Then we just have to impose the extra condition that the other Sylow subgroups are cyclic. This implies in particular that $O(G)$ is metacyclic.

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Let p be a monotone graph parameter (that is, adding edges to a graph cannot decrease the value of the parameter). Now for each consecutive pair of graphs in the hierarchy, we can ask: for which groups, do the values of p on the corresponding graphs coincide?
There are plenty of open questions here; the only case to have been looked at (as far as I know) is the power graph and enhanced power graph. Again not many results are known. Recall that these graphs are equal for a group G if and only if every element of G has prime power order.

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One slightly surprising thing about the second result is that we do not have a formula for the matching number of $\text{Pow}(G)$ for an arbitrary group G. The theorem is proved by showing that, given any matching in $\text{EPow}(G)$, we can find another matching of the same size which has fewer edges which don't belong to $\text{Pow}(G)$.

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So our problem includes the determination of all Fermat and Mersenne primes!

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I will look at one further property to illustrate the benefit of treating the graphs as a hierarchy.

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If Γ is the comparability graph of a finite partial order, then there is a finite group G such that Γ is isomorphic to an induced subgraph of $\mathrm{Pow}(G)$.

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But using our hierarchy, we can strengthen the last result.

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Suppose that the edges of a finite complete graph are coloured red, green and blue in any manner. Then the vertex set can be embedded into a finite group G such that

- *the red edges belong to* $\mathrm{EPow}(G)$;
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- ignoring the red-green distinction, commuting graphs form a universal class;
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The independence graph has an edge $\{x, y\}$ whenever $\{x, y\}$ is a subset of a minimal (with respect to inclusion) generating set of G; the rank independence graph has an edge $\{x, y\}$ whenever $\{x, y\}$ is a subset of a generating set of minimum cardinality $d(G)$.

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When do these implications reverse?

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In each case, the groups can be classified.

Clique number of the power graph

As a final topic, there is a sense in which the enhanced power graph is not much larger than the power graph. For example, while $\omega(\mathrm{Pow}(G)) \leq \omega(\mathrm{EPow}(G))$, it is true the $\omega(\mathrm{EPow}(G))$ is bounded above by a function of $\omega(\mathrm{Pow}(G))$. This can be seen by looking more closely at the clique number of $\mathrm{Pow}(G)$.

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Any edge of $\mathrm{Pow}(G)$ is contained in a cyclic subgroup; and if every pair of vertices of a set S in a group are contained in a cyclic subgroup, then S is contained in a cyclic subgroup. So $\omega(G)$ is equal to the maximum of $\omega(C)$ over all cyclic subgroups C of G.

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Similarly, $\omega(\mathrm{EPow}(G))$ is equal to the order of the largest cyclic subgroup of G.

So it suffices to look at cyclic groups.

In a cyclic group

Let $f(n)$ be the clique number of $\mathrm{Pow}(C_n)$, where C_n is the cyclic group of order n.

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From this it follows easily that $f(n) \leq 3\phi(n)$. Hence n is bounded above by $cm \log\log m$, where $m = f(n)$; and the same bound holds for the clique numbers m and n of the power graph and enhanced power graph of an arbitrary group.

In fact,

$$\limsup f(n)/\phi(n) = 2.6481017597\ldots,$$

where the constant on the right is

$$\sum_{k\geq 0} \prod_{i=1}^{k} \frac{1}{p_i - 1},$$

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This suggests several questions, such as

- is this constant rational, algebraic or transcendental?
- what other numbers are limit points of the set $\{f(n)/\phi(n) : n \in \mathbb{N}\}$?

- Peter J. Cameron, Graphs defined on groups, *International Journal of Group Theory* **11** (2022), 43–124; doi: 10.22108/ijgt.2021.127679.1681

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… for your attention.