

Graphs defined on groups

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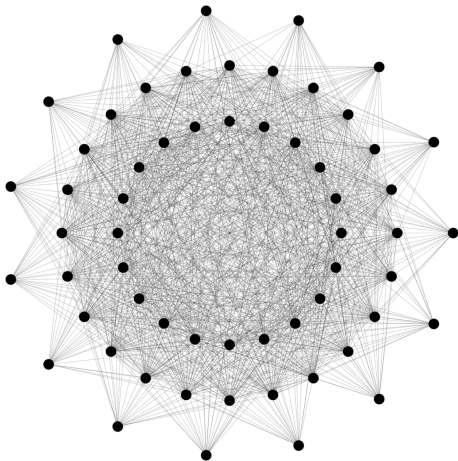
Graphs and Groups Seminar
24/25 March 2021

These are the slides of two talks I gave to the Graphs and Groups research seminar run from CUSAT in Kochi, Kerala, India on 24 and 25 March 2021.

The changes are:

- ▶ I have corrected a couple of misprints.
- ▶ There is more material here than I covered in the lectures.
- ▶ The theorem that the power graph of a group of prime power order is a cograph was not in my lectures, but was discussed afterwards, so I have included the proof.

My thanks to the audience for their comments.



The generating graph of A_5 (with thanks to Scott Harper)

Vertices are non-identity elements of $G = A_5$

x is joined to y if $\langle x, y \rangle = G$.

Graphs and groups

I am delighted to be giving two talks to open this seminar. It is an auspicious occasion!

“Graphs and groups” is a very big subject, taking in geometric group theory, much of algebraic graph theory, aspects of computation such as the **graph isomorphism problem**, finite and infinite permutation groups, and much else besides. But my goals are more limited.

In these two talks I will be speaking mainly about the following situation. G is a group, usually finite but it may be infinite; Γ is a graph with vertex set G (or some subset of G), whose edges reflect in some way the group structure of G .

Typical examples, which you may have met, include the **commuting graph** and the **power graph**.

Outline

In these two talks I hope to

- ▶ introduce a hierarchy of graphs;
- ▶ describe a few of their properties;
- ▶ develop some graph theory to help study them;
- ▶ relate them to other graphs (Gruenberg–Kegel graph, intersection graphs);
- ▶ pose a number of open problems;
- ▶ provide you with some tools to study these problems.

One particular class of problems concerns **cographs**. I will define these, show why they are important for our problem, and pose the question: for which groups G is the power graph (or commuting graph, or one of the others) a cograph?

Much more detail, and many references, can be found in the preprint: arXiv 2102.11177.

Notation

A talk about groups typically begins “Let G be a group ...”, while a talk about graphs will start “Let G be a graph ...”. We will be talking about both, so we have to make a decision. “Graph” is a Greek word, so it makes sense for a graph to be Γ . “Group” is a German word, so perhaps a group should be \mathfrak{G} ; but I never learned how to do a Gothic G in handwriting, and probably you didn’t either, so I will use G for a group.

Notation, 2

Various *ad hoc* notations have been used for particular kinds of graphs defined on a group G , such as $\mathcal{G}(G)$ or $\Gamma(G)$. Since I will be talking about several kinds of graphs, this will not really work either.

So my notation will be like this: the power graph of G will be $\text{Pow}(G)$, the commuting graph will be $\text{Com}(G)$. Similar notation will be introduced for other kinds of graphs, when I describe these, as I will do next.

For easy comparison, I will simply define each of these graphs to have vertex set the whole of G . For some questions it makes sense to talk about the “reduced” graph, where the identity, or the elements in the centre, or perhaps the vertices joined to all others, are removed.

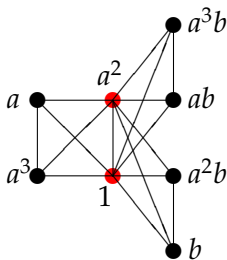
Apart from this, most of my notation for graphs and groups will be standard.

Dramatis Personae, 1: the commuting graph

The commuting graph was the first of the four to be defined; it appeared in the famous paper of Brauer and Fowler on centralisers of involutions in finite groups in 1955.

The **commuting graph** $\text{Com}(G)$ of G has vertex set G ; vertices g and h are joined if and only if $gh = hg$. (This definition would put a loop at every vertex; we silently suppress these.)

Here are the commuting graphs of the two non-abelian groups of order 8: $D_8 = \langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$.



Dramatis Personae, 2: the power graph

Next on the scene, much later, was the power graph, defined by Kelarev and Quinn in 1999.

The **directed power graph** $\text{DPow}(G)$ of G has vertex set G ; there is an arc from g to h if h is a power of g . Again we ignore loops. The **power graph** $\text{Pow}(G)$ of G is obtained by ignoring the directions on the arcs (and suppressing one of them if each of g and h is a power of the other); in other words, g and h are joined if one is a power of the other.

The power graph does not uniquely determine the directions; but the following holds:

Theorem

If two groups have isomorphic power graphs, then they have isomorphic directed power graphs.

The graph shown earlier is the power graph of Q_8 , but not of D_8 .

Dramatis Personae, 3: the enhanced power graph

The enhanced power graph is more recent. The complementary graph was defined by Abdollahi and Hassanabadi in 2007 under the name “noncyclic graph”; I will use the description appearing in a paper of Aalipour *et al.* in 2017.

The **enhanced power graph** $\text{EPow}(G)$ has vertex set G ; we join g to h if and only if there is an element k such that both g and h are powers of k . Equivalently, g and h are joined if and only if $\langle g, h \rangle$ is cyclic.

We saw that the power graph determines (up to isomorphism) the directed power graph, and hence the enhanced power graph. The converse is also true; if two graphs have isomorphic enhanced power graphs, then they have isomorphic power graphs.

Dramatis Personae, 4: the deep commuting graph

The final actor, the deep commuting graph, is not yet published, but can be found in a paper by Bojan Kuzma and me on the arXiv.

The deep commuting graph $\text{DCom}(G)$ has vertex set G ; vertices g and h are joined if and only if their inverse images in every central extension of G commute. (That is, if $Z \leq Z(H)$ with $H/Z \cong G$, and aZ and bZ are the cosets of Z corresponding to g and h , then we require that a and b commute (in every such extension).

We showed that it suffices to take a single central extension of G , namely a **Schur cover** (where $Z \leq Z(H) \cap H'$ and subject to this Z is as large as possible). It is independent of the choice of Schur cover.

For example, the Klein group of order 4 has two Schur covers, the dihedral and quaternion groups of order 8; so the deep commuting graph of V_4 is the star $K_{1,3}$.

The deep commuting graph is harder to work with than the others.

Dramatis Personae, 5: the non-generating graph

The **generating graph** $\text{Gen}(G)$ of a group G is the graph with vertex set G in which a and b are joined if and only if $G = \langle a, b \rangle$. (If G cannot be generated by two elements, this is the graph with no edges.) The **non-generating graph** $\text{NGen}(G)$ is the complement of the generating graph.

It follows from the **Classification of Finite Simple Groups** that every non-abelian finite simple group can be generated by two elements; so we have some interesting examples to play with! There are several other graphs which have been studied, some of which you can find in my manuscript mentioned earlier; but that will do for now.

Different behaviour

The power graph, enhanced power graph, and commuting graph all have the following nice property:

The induced subgraph of $X(G)$ on a subgroup H of G is $X(H)$.

It is quite different for the non-generating graph. The induced subgraph of $\text{NGen}(G)$ on a proper subgroup H of G is the complete graph on H , since two elements of H cannot generate G .

The deep commuting graph is much more irregular, caused by the strange behaviour of the Schur multiplier.

The hierarchy

The first four graphs defined above on G form a hierarchy, in the order power graph, enhanced power graph, deep commuting graph, commuting graph, in the sense that the edge set of each is contained in the edge set of the next.

This is clear in all cases except the enhanced power graph and deep commuting graph. So suppose that $\{g, h\}$ is an edge of the enhanced power graph, so that $\langle g, h \rangle = \langle k \rangle$ for some k . Now let H be a central extension of G with kernel Z . Let a, b, c be representatives of the cosets of Z in H corresponding to g, h, k . Then $\langle Z, a, b \rangle = \langle Z, c \rangle$, which (as a cyclic extension of a central subgroup) is abelian; so a and b commute.

Furthermore, if G is non-abelian or not 2-generated, then the edge set of the commuting graph is contained in the edge set of the non-generating graph, so the hierarchy has one more term. We can also put the complete graph at the top and the null graph at the bottom.

Equality

Now we have built such a hierarchy, we can ask: when are two of these graphs equal?

We have at least partial answers in most cases:

- ▶ $\text{Pow}(G)$ is equal to the null graph if and only if G is the trivial group.
- ▶ $\text{EPow}(G) = \text{Pow}(G)$ if and only if G contains no subgroup $C_p \times C_q$, for distinct primes p and q .
- ▶ $\text{Com}(G) = \text{EPow}(G)$ if and only if G contains no subgroup $C_p \times C_p$, for prime p .
- ▶ For G nonabelian, $\text{NGen}(G) = \text{Com}(G)$ if and only if G is a **minimal non-abelian group**.
- ▶ $\text{NGen}(G)$ is complete if and only if G cannot be generated by two elements.

At least something is known in all these cases, except the last where there is probably nothing to say.

Groups not containing $C_p \times C_q$ are those in which every element has prime power order (these are sometimes called **EPPO groups**). There is a classification in my paper with Natalia Maslova.

Groups containing no $C_p \times C_p$ are those in which the Sylow subgroups are cyclic or generalized quaternion. Known classification theorems determine these groups.

Minimal non-abelian groups were determined by Miller and Moreno back in 1904.

What about the deep commuting graph?

The deep commuting graph was introduced specifically as a graph lying between the enhanced power graph and the commuting graph. So when can it be equal to either of these? The answer is not so easy to state. Equality of the enhanced power graph and the deep commuting graph depends on a group-theoretic condition; but equality of the deep commuting graph and the commuting graph involves yet another construction, the **Bogomolov multiplier** of a group. (These two graphs on G are equal if and only if the Bogomolov multiplier is equal to the Schur multiplier.)

I don't understand the Bogomolov multiplier very well; but it is always trivial if G is a non-abelian simple group, and can be computed with the GAP package HAP.

Differences

If two graphs in the hierarchy are not equal, we can ask about their difference, the graph containing the edges of the larger which are not edges of the smaller.

The difference between the power graph and the null graph is the power graph, and the difference between the complete graph and the non-generating graph is the generating graph. Both of these have been intensively studied.

To my knowledge, only one other case has been looked at. This is the difference between the non-generating graph and the commuting graph, for non-abelian groups. Saul Freedman, a PhD student at the University of St Andrews, has considered these; his thesis will contain detailed results about the question of connectedness of these graphs.

A problem

Other cases are open. So let us take what is probably the easiest case. Given a group G , consider the graph where x is joined to y if and only if they are joined in the commuting graph but not in the power graph (in other words, $xy = yx$ but neither of x and y is a power of the other).

Question

Choose your favourite graph-theoretic property or parameter, and study this graph on a group G : when does it have your property, or what is the value of your parameter, on that graph?

If you succeed, carry on with some of the other differences!

Universality

Of course, this problem can be formulated also for the original graphs in the hierarchy. Here, of course, much more is known: there have been detailed study of the commuting graph and the power graph, and work has begun on the enhanced power graph. Here is a sample result.

Theorem

- ▶ *The power graph of any finite group is the comparability graph of a partial order; but every finite graph which is the comparability graph of a partial order is an induced subgraph of the power graph of some group.*
- ▶ *Each of the other non-trivial graph types in the hierarchy is **universal**: that is, every finite graph is an induced subgraph of the enhanced power graph (or commuting graph, or ...) of some finite group.*

Question

- ▶ Find a good upper bound for the order of a group G such that every n -vertex graph is an induced subgraph of the power graph (or commuting graph, or ...) of G .
- ▶ Find a good upper bound for the function f such that every n -vertex graph is an induced subgraph of the power graph (or commuting graph, or ...) of a group of order at most n .

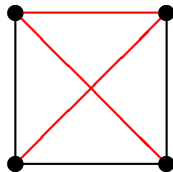
For the second question, for the commuting graph the function $f(n) = 2^{n+1}$ suffices. There are less good bounds in the other cases.

For the first, the best bound I have is 2^{2n+1} : we can take G to be an extraspecial group of order 2^{2n+1} (the central product of n copies of the quaternion group of order 8).

Plenty of good problems here!

Cographs

Now I am going to tell you about cographs. You will not see at first why they are an important class of graphs, but they are! The fact that they have been rediscovered several times with different names (complement-reducible graphs, hereditary Dacey graphs, N-free graphs) should indicate their significance. A finite graph Γ is a **cograph** if it has no induced subgraph isomorphic to the 4-vertex path. Since the 4-vertex path graph is self-complementary, the complement of a cograph is a cograph.



Properties of cographs

Here are a few properties of cographs:

Theorem

- ▶ *If a cograph on more than one vertex is connected, then its complement is disconnected.*
- ▶ *Cographs form the smallest non-empty class of graphs which is closed under the operations of complement and disjoint union; so every cograph can be built from the 1-vertex graph by these operations.*
- ▶ *The automorphism group of a cograph can be built from the trivial group by the operations “direct product” and “wreath product with a symmetric group”.*

The proofs are not too difficult, and are left as an exercise. The recursive structure gives efficient algorithms for many problems on cographs which are hard for general graphs.

Twins



Two vertices in a finite graph Γ are **twins** if they have the same neighbours (possibly excepting one another).

Twins can be interchanged by an automorphism fixing all other vertices.

There are two kinds of twins. If I need to distinguish, I will say that x and y are **open twins** if they have the same open neighbourhoods, and are **closed twins** if they have the same closed neighbourhoods. Note that open twins are not joined, whereas closed twins are joined.

Being twins is an equivalence relation. In each equivalence class, all pairs of vertices are of the same type (open twins, or closed twins).

Twin reduction

If we collapse a pair of twins to a single vertex, and continue this until there are no more pairs of twins, the result (which I will call the **cokernel** of the graph) is independent of the sequence of reductions.

Theorem

Γ is a cograph if and only if its cokernel is the 1-vertex graph.

Proof.

First show that a cograph on more than 1 vertex has twins. The proof is by induction. If the graph is null, it is clear; if it is disconnected but not null, a nontrivial component has twins; and if it is connected, consider the complement. Moreover, twin reduction preserves the cograph property.

Conversely, an induced 4-vertex path has no twins and so cannot be destroyed by twin reduction. □

Forbidden subgraphs

Given a graph F , let $\text{Forb}(F)$ denote the class of graphs containing no induced subgraph isomorphic to F . Suppose that F has no pairs of twins. Then twin reduction can neither create nor destroy an induced copy of F . Hence

Theorem

Let F be a twin-free graph. Then a graph belongs to $\text{Forb}(F)$ if and only if its cokernel does.

Perfect graphs

A graph is **perfect** if every induced subgraph has clique number equal to chromatic number. According to the **Strong Perfect Graph Theorem** of Chudnovsky *et al.*, a graph is perfect if and only if it has no induced subgraph which is an odd cycle on at least five vertices or the complement of one. Since all these graphs are twin-free, we conclude:

Proposition

A graph is perfect if and only if its cokernel is perfect.

In particular, cographs are perfect (a fact which is easily proved directly).

Power graphs are perfect

As a small diversion, I show:

Theorem

The power graph of a finite group is a perfect graph.

Proof.

This is because the directed power graph is a **partial preorder** (a reflexive and transitive relation), and the power graph is its **comparability graph**. It is easy to show that we can replace any partial preorder by a partial order with the same comparability graph. Now the comparability graph of a partial order is perfect, by Dilworth's Theorem (the easy direction). \square

For the other types of graph, we do not know for which groups they are perfect.

Graphs in the hierarchy always have twins

Theorem

For each of the types X of graph introduced earlier, if G is a non-trivial group, then $X(G)$ has non-trivial twin relation.

Proof.

If g is an element of G with order $m > 2$, and $\gcd(m, d) = 1$, then g and g^d are closed twins (i.e. have the same closed neighbourhood).

The only case remaining is that when G is an elementary abelian 2-group, in which case $X(G)$ is either complete (for the commuting graph, or the non-generating graph if the order is greater than 4) or a star (for the others). □

Automorphisms

If x and y are twins, then the transposition of x and y , fixing all other vertices, is an automorphism of the graph. So its automorphism group has a normal subgroup which is the direct product of symmetric groups on the twin classes.

So if you are interested in the automorphism group, you might first want to carry out twin reduction. Of course, if the graph is a cograph, the result will be the trivial graph ...

The first time I thought about the non-generating graph of A_5 , I was shocked to find that its automorphism group has order 23482733690880. Most of this is accounted for by the normal subgroup $(S_2)^{10} \times (S_4)^6$. In fact the cokernel has 32 vertices, and its automorphism group is $\text{Aut}(A_5) = S_5$.

When is the power graph a cograph?

For any of the four types of graph in our hierarchy, and any group G of order greater than 1, the corresponding graph on G has non-trivial twin relation. For, if g is an element of order $m \geq 2$, and d is coprime to m , then g and g^d are twins. In the excluded case, G is an elementary abelian 2-group; this is easily dealt with.

So we can apply twin reduction, and it is of some interest to know whether we reach the 1-vertex graph (in other words, whether the graph is a cograph).

I will say a few words about how this problem might be attacked, starting with the power graph.

Connectedness

Whether a graph is a cograph is unaffected by adding isolated vertices, or vertices joined to all others.

For the power graph, we know the set of vertices joined to all others:

- ▶ if G is cyclic of prime power order, the whole of G (the power graph is complete);
- ▶ if G is cyclic but not of prime power order, the set consisting of the identity and the generators;
- ▶ if G is generalized quaternion, $Z(G)$;
- ▶ in all other cases, just the identity.

So these are the isolated vertices in the complement of the power graph.

Theorem

For any finite group G , the complement of the power graph has a single connected component, apart from isolated vertices.

The upshot of this theorem is that, if $\text{Pow}(G)$ is a cograph, then after removing isolated vertices, it is disconnected.

Using the fact that the complement of the power graph is obtained from the complement of the commuting graph by adding edges, this theorem follows fairly easily from the following:

Theorem

For any finite group G , the complement of the commuting graph has a single connected component, apart from isolated vertices. This component has diameter at most 2.

Proof.

In the commuting graph $\text{Com}(G)$, the closed neighbourhood of any vertex g is a subgroup of G , namely the **centraliser** $C_G(g)$. If $g \in Z(G)$, then it is joined to all other vertices in $\text{Com}(G)$, and hence is isolated in the complement.

Suppose that neither g nor h is in $Z(G)$. Then $C_G(g)$ and $C_G(h)$ are proper subgroups of G . Since a finite group cannot be written as a union of two proper subgroups (this is an exercise for you if you haven't seen it before), we see there is an element k outside both. Then k is joined to g and h in the complement of $\text{Com}(G)$. □

Nilpotent groups

A finite group is **nilpotent** if it is the direct product of its Sylow subgroups.

Last year, Pallabi Manna, Ranjit Mehatari and I proved the following.

Theorem

Let G be a nilpotent group whose power graph is a cograph. Then either G is a p -group for some prime p , or G is cyclic of order pq where p and q are distinct primes.

The point of this theorem is that it tells us that, if the power graph of an arbitrary group G is a cograph, then any nilpotent subgroup of G apart from its Sylow p -subgroups must be cyclic of order pq for distinct primes p and q .

The converse

This was not in the talk but arose in the discussion so I have included a proof.

Theorem

If the group G has prime power order, then its power graph is a cograph.

Proof.

Let (a, b, c) be a 3-vertex induced path. In the directed power graph, we cannot have $a \rightarrow b \rightarrow c$ or $c \rightarrow b \rightarrow a$, since \rightarrow is transitive; and we cannot have $b \rightarrow a$ and $b \rightarrow c$, since then a and c lie in the cyclic group $\langle b \rangle$ of prime power order, and so one is a power of the other. So we have $a \rightarrow b$ and $c \rightarrow b$.

If (a, b, c, d) is a 4-vertex path, then $c \rightarrow b$ and (looking at the path (b, c, d)) also $b \rightarrow c$; so b and c generate the same cyclic group, contradicting the fact that a is joined to b but not to c . \square

A test problem

The next problem is maybe not so interesting in itself, but could be a good indicator of how well we understand groups whose power graph is a cograph.

Question

Is the following true? Let Γ be a cograph. Then there exists a group G such that $\text{Pow}(G)$ is a cograph and contains Γ as an induced subgraph.

This is tricky for a couple of reasons:

- ▶ The inductive scheme for cographs depends on taking the complement. But it is not clear what the complement of a power graph is.
- ▶ The result above about nilpotent groups implies that there will be no nice direct product construction to handle disjoint unions.

Probably we have to look for a p -group G .

What happens for $\text{PSL}(2, q)$?

Let $q = p^a$, p prime. Let l and m be $q - 1$ and $q + 1$ if q is even, or $(q - 1)/2$ and $(q + 1)/2$ if q is odd. Then, apart from the (abelian) Sylow p -subgroup, the maximal abelian subgroups of $\text{PSL}(2, q)$ are cyclic groups of orders l and m , whose normalisers are dihedral (hence not nilpotent unless l or m is a power of 2). So we have:

Theorem

Let l and m be as above. Then the power graph of $G = \text{PSL}(2, q)$ is a cograph if and only if each of l and m is either a prime power or a product of two distinct primes.

Question

Are there infinitely many prime powers q for which the power graph of $\text{PSL}(2, q)$ is a cograph?

This is probably quite a hard number-theoretic problem. It has a superficial resemblance to questions about Fermat and Mersenne primes, and about the Catalan conjecture on proper powers differing by 1 (now the theorem of Mihăilescu), but is not equivalent to any of these.

The values of d up to 200 for which the conditions of the theorem hold for $q = 2^d$ are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.

Small odd prime powers with the property are easily listed.

The first few are 3, 5, 7, 9, 11, 13, 17, 19, 27, 29, 31, 43, 53, 67, 163, 173, 243, 257, 283, 317

Other simple groups

The table on the next slide gives the sizes of cokernels of various graphs defined on the first few finite non-abelian simple groups G by order. The columns give the power graph, enhanced power graph, deep commuting graph, commuting graph, and non-generating graph.

Recall that a graph is a cograph if and only if its cokernel has 1 vertex.

Question

For the various types X of graph, for which simple groups G is $X(G)$ a cograph?

| G | $ G $ | Pow | EPow | DCom | Com | NGen |
|---------------------|-------|------|------|------|------|------|
| A_5 | 60 | 1 | 1 | 1 | 1 | 32 |
| $\text{PSL}(2, 7)$ | 168 | 1 | 1 | 1 | 44 | 79 |
| A_6 | 360 | 1 | 1 | 1 | 92 | 167 |
| $\text{PSL}(2, 8)$ | 504 | 1 | 1 | 1 | 1 | 128 |
| $\text{PSL}(2, 11)$ | 660 | 1 | 1 | 1 | 112 | 244 |
| $\text{PSL}(2, 13)$ | 1092 | 1 | 1 | 1 | 184 | 366 |
| $\text{PSL}(2, 17)$ | 2448 | 1 | 1 | 1 | 308 | 750 |
| A_7 | 2520 | 352 | 352 | 352 | 352 | 842 |
| $\text{PSL}(2, 19)$ | 3420 | 1 | 1 | 1 | 344 | 914 |
| $\text{PSL}(2, 16)$ | 4080 | 1 | 1 | 1 | 1 | 784 |
| $\text{PSL}(3, 3)$ | 5616 | 756 | 756 | 808 | 808 | 1562 |
| $\text{PSU}(3, 3)$ | 6048 | 786 | 534 | 499 | 499 | 1346 |
| $\text{PSL}(2, 23)$ | 6072 | 1267 | 1 | 1 | 508 | 1313 |
| $\text{PSL}(2, 25)$ | 7800 | 1627 | 1 | 1 | 652 | 1757 |
| M_{11} | 7920 | 1212 | 1212 | 1212 | 1212 | 2444 |

The table suggests various conjectures, some of which can be proved. For example:

Theorem

The non-generating graph of a non-abelian finite simple group is not a cograph.

Proof.

Recently Burness, Guralnick and Harper have shown that the reduced generating graph of a finite simple group has **spread** (at least) 2: this means that any two non-identity elements have a common non-identity neighbour. So the generating graph is connected.

Also Shen proved that the reduced non-generating graph of a finite simple group is connected; recently Freedman showed that the best bound for the diameter is either 4 or 5.

To finish the proof, recall that if a cograph is connected, then its complement is disconnected; and adding a vertex joined to all others does not change the property of being a cograph. □

Question

Find, or estimate, the number of vertices in the cokernel of the non-generating graph of a finite simple group (or indeed, of some of the other types of graph we are considering).

It is possible that results on the generating and non-generating graphs will give some information about this.

The commuting graph

Question

For which groups G is the commuting graph of G (a) a cograph, (b) perfect?

We have another tool to help us answer this question. The **strong product** $\Gamma \boxtimes \Delta$ of graphs Γ and Δ is the graph whose vertex set is the Cartesian product of the vertex sets of Γ and Δ , with vertices (v_1, w_1) and (v_2, w_2) joined whenever v_1 is equal or adjacent to v_2 and w_1 is equal or adjacent to w_2 , but not equality in both places. (All of the graphs in the hierarchy naturally have loops at each vertex, which we have discarded; the strong product is the natural categorical product in the category of graphs with a loop at each vertex.)

The strong product, along with the Cartesian and categorical products, is denoted by a symbol representing the corresponding product of two edges: the Cartesian product is $\Gamma \square \Delta$, while the categorical product is $\Gamma \times \Delta$.

Direct products

Theorem

Let G and H be finite groups.

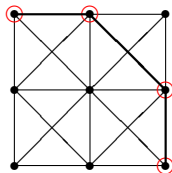
- ▶ $\text{Com}(G \times H) = \text{Com}(G) \boxtimes \text{Com}(H)$.
- ▶ If G and H have coprime orders, then $\text{EPow}(G \times H) = \text{EPow}(G) \boxtimes \text{EPow}(H)$.
- ▶ If G/G' and H/H' have coprime orders, and in particular if G and H are perfect groups, then $\text{DCom}(G \times H) = \text{DCom}(G) \boxtimes \text{DCom}(H)$.

Thus, questions about the commuting graph or enhanced power graph of a nilpotent group can be reduced to questions about the corresponding graphs for their Sylow subgroups.

Perfectness

To help with these questions, I note that the perfectness of the strong product of graphs has been studied by Ravindra and Parthasarathy. So we should be able to decide which nilpotent groups have perfect commuting graphs or enhanced power graphs.

Cographs will be more difficult, since the class of cographs is not preserved by the strong product:



Question

When is the strong product of cographs a cograph?

The Gruenberg–Kegel graph

The **Gruenberg–Kegel graph** (sometimes called the **prime graph**) of a finite group G is defined as follows. Its vertex set is the set of prime divisors of the order of G ; an edge joins p to q if and only if G contains an element of order pq .

These graphs were introduced by Gruenberg and Kegel in an unpublished manuscript concerned with the integral group ring of G , and in particular the decomposability of the augmentation ideal.

Their main result (published later by Gruenberg's student Williams) is a structure theorem for groups whose Gruenberg–Kegel graph is disconnected. It asserts that such a group is either

- ▶ a **Frobenius** or **2-Frobenius** group; or
- ▶ an extension of a nilpotent π -group by a simple group by a π -group, where π is the set of primes in the connected component containing 2.

Applications of the GK graph

Theorem

If two groups have isomorphic power graphs (or enhanced power graphs, or deep commuting graphs, or commuting graphs), then they have the same Gruenberg–Kegel graph.

Theorem

Let G be a group with $Z(G) = 1$. Then the reduced commuting graph (the induced subgraph of the commuting graph on the set of non-identity elements) is connected if and only if the Gruenberg–Kegel graph of G is connected.

Theorem

Let G be a finite group. Then $\text{Pow}(G) = \text{EPow}(G)$ if and only if the Gruenberg–Kegel graph of G is a null graph (a graph with no edges).

These groups were mentioned earlier under the name **EPPO groups**.

The GK graph and cographs

It also follows that, if the GK graph of G has no edges (that is, if G is an EPPO group), then $\text{Pow}(G)$ is a cograph.

The converse is false; but from Williams' results on the GK graph and my result with Manna and Mehatari on nilpotent groups, we get the following:

Theorem

Suppose that G is a finite non-solvable group for which $\text{Pow}(G)$ is a cograph. Then each connected component of the Gruenberg–Kegel graph of G has at most two vertices, except possibly for the component containing the prime 2.

The GK graph does not completely determine whether $\text{Pow}(G)$ is a cograph. Consider the two simple groups $\text{PSL}(2, 11)$ and M_{11} . They have the same GK graph (it has four vertices $\{2, 3, 5, 11\}$; there is an edge $\{2, 3\}$ while 5 and 11 are isolated), but the power graph of the first group is a cograph, while that of the second is not.

A duality on graphs

In order to study intersection graphs, we need to define a “duality” relation on graphs.

Let B be a bipartite graph (with a specified bipartition (V_1, V_2)). Let Γ_1 and Γ_2 be the graphs whose vertex sets are V_1 and V_2 , two vertices adjacent if they lie at distance 2 in B . These are the **halved graphs** of B .

We call a pair of graphs Γ_1 and Γ_2 a **dual pair** if there is a bipartite graph B without isolated vertices such that Γ_1 and Γ_2 are the halved graphs of B .

Theorem

Let Γ_1 and Γ_2 be a dual pair of graphs. Then there is a bijection between the connected components of Γ_1 and those of Γ_2 such that the diameters of corresponding components are equal or differ by 1.

Intersection graphs

The **intersection graph** of a non-trivial group G is the graph whose vertices are the non-trivial proper subgroups of G , two vertices adjacent if the corresponding subgroups have non-trivial intersection.

Theorem

The reduced non-generating graph of G and the intersection graph of G form a dual pair.

Proof.

Let B be the bipartite graph whose vertices are the non-identity elements of G and non-trivial proper subgroups of G , with g joined to H if $g \in H$. Then B witnesses that the two graphs of the theorem form a dual pair. \square

So these two graphs have the same number of connected components, and the diameters of corresponding components differ by at most 1.

Other examples

A simple modification of this argument gives us more examples of dual pairs:

- ▶ the commuting graph of G and the intersection graph of non-trivial abelian proper subgroups of G ;
- ▶ the enhanced power graph of G and the intersection graph of non-trivial cyclic proper subgroups of G .

Question

Do other properties apart from connected components transfer between the members of a dual pair?

Other graphs on a group

Other graphs which can be considered include

- ▶ the **nilpotency graph** of G , with $x \sim y$ if $\langle x, y \rangle$ is nilpotent;
- ▶ the **solvability graph** of G , with $x \sim y$ if $\langle x, y \rangle$ is solvable;
- ▶ the **directed Engel graph** of G , with $x \rightarrow y$ if $[y, x, x, \dots, x] = 1$ (with an arbitrary number of x s), or its undirected version.

For many groups, including non-abelian finite simple groups, the nilpotency and solvability graphs fit into the hierarchy above the commuting graph but below the non-generating graph.

Also, all these graphs can be defined on infinite groups; quite a bit is known.

But time does not permit discussion of this here.

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