Progress resulting from RDGG

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Research Discussion on Graphs and Groups
11 August 2021
(with contributions from several participants)

Introduction

I would like to begin with a sincere thank-you to those involved in organising the RDGG, especially Vijay for setting it up and asking me to contribute, and Aparna for her invaluable support.
Vijay said to me that he had acted according to the precept in the *Bhagavad-Gita*: “To action alone one has the right but not to its fruits.” He saw my paper on the arXiv and acted. But in agreement with the way this precept works, we are now all enjoying the fruits of his action.

As a result of this discussion group, I have had the opportunity of working with many colleagues from all over India and beyond, which has been for me a wonderful experience. So thank you all, especially G. Arunkumar, V. Arvind, T. Tamizh Chelvam, Angsuman Das, Saul Freedman, T. Kavaskar, Ranjit Mehatari, Lavanya Selvaganesh, and Swathi V. V.

I want to try to summarise some of the progress we have made. The picture on the cover page suggests the beautiful country we have begun to explore; there are more good things to find.

Summary

I will report progress on the following topics:

- When is the power graph a cograph?
- Clique number of the power graph and a new constant
- Matching number of the power graph
- A two-dimensional extension of the hierarchy
- Are zero-divisor graphs of local rings universal?

But I stress that this is not a complete account of research arising from RDGG. More is on the way!

The hierarchy

I will be mainly discussing the hierarchy, but let me briefly remind you of the graphs on G, with the rule for joining x to y:

- $\mathrm{Pow}(G)$, the power graph: one of x and y is a power of the other;
- $\mathrm{EPow}(G)$, the enhanced power graph: $\langle x, y\rangle$ is cyclic;
- $\mathrm{DCom}(G)$, the deep commuting graph: the inverse images of x and y commute in every central extension of G;
- $\mathrm{Com}(G)$, the commuting graph: $xy = yx$;
- $\mathrm{NGen}(G)$, the nongenerating graph: $\langle x, y\rangle \neq G$.

The first four form a hierarchy under inclusion of edge sets, and the fifth is at the top if G is either non-abelian or not 2-generated.

When is the power graph of G a cograph?

A cograph is a graph containing no induced subgraph isomorphic to the 4-vertex path.
Note that the class of groups whose power graph is a cograph is subgroup-closed.
This was one of the questions I asked in my introductory lectures. At that stage, Pallabi Manna, Ranjit Mehatari and I had determined all the nilpotent groups with this property: groups of prime power order, and cyclic groups whose order is the product of two distinct primes.
The paper has just been published:

- Pallabi Manna, Peter J. Cameron and Ranjit Mehatari, Forbidden subgraphs of power graphs, *Electronic J. Combinatorics* **28(3)** (2021), Paper #P3.4

This result has been the key to further developments.

A companion problem

An EPPO group is one in which every element has prime power order.

Theorem

For a finite group G, the following are equivalent:

- *G is an EPPO group;*
- *the power graph of G is equal to the enhanced power graph;*
- *the Gruenberg–Kegel graph of G is a null graph.*

Groups having these equivalent properties also have the property that their power graphs are cographs.

These groups were investigated by Higman in 1957. By 1962, Suzuki had found all the simple EPPO groups; but the complete determination is very recent, by Natalia Maslova and me, and not yet published.

Manna, Mehatari and I have succeeded in determining all non-abelian finite simple groups whose power graph is a cograph, up to some hard number theoretic problems.

- $\mathrm{PSL}(2,q)$, with q a prime power. If q is a power of 2, put $\{l,m\} = \{q-1, q+1\}$; if it is an odd prime power, put $\{l,m\} = \{(q-1)/2, (q+1)/2\}$.
  Then the power graph of $\mathrm{PSL}(2,q)$ is a cograph if and only if each of l and m is either a prime power or the product of two distinct primes.
- $\mathrm{Sz}(q)$, with q an odd power of 2 (greater than 2). Let $\{k,l,m\} = \{q-1, q+\sqrt{2q}+1, q-\sqrt{2q}+1\}$.
  Then the power graph of $\mathrm{Sz}(q)$ is a cograph if and only if each of k, l and m is either a prime power or the product of two distinct primes.

Apart from these, there is only one further non-abelian simple group whose power graph is a cograph: $\mathrm{PSL}(3,4)$.
The necessity of the conditions for $\mathrm{PSL}(2,q)$ and $\mathrm{Sz}(q)$ is because these groups have cyclic subgroups of orders (k), l and m. The sufficiency, and the analysis of the remaining simple groups, requires detailed knowledge of the finite simple groups and their subgroups.
However, the story of EPPO groups suggests that finishing the job may still be some way off.

The clique number of the power graph

A formula for the clique number (the size of the largest complete subgraph) of the power graph of a finite group G was found by Alireza, Erfanian and Abbas.

In reworking their proof, I have (I think) added something interesting.

Since the power graph is a spanning subgraph of the enhanced power graph, we have $\omega(\mathrm{Pow}(G)) \leq \omega(\mathrm{EPow}(G))$. In my paper in the *International Journal of Group Theory*, I showed that there is an inequality in the other direction: there is a function F such that $\omega(\mathrm{EPow}(G)) \leq F(\omega(\mathrm{Pow}(G)))$. I asked for the best possible function. The function I gave there was exponential, but the correct value is a little faster than linear (i.e. $O(n \log \log n)$).

Cyclic groups

Any edge of the power graph of a group is contained in a cyclic subgroup of the group. A set of group elements with the property that any two generate a cyclic group must be contained in a cyclic subgroup. So any clique is contained in a cyclic subgroup, and we have

$$\omega(\mathrm{Pow}(G)) = \max\{\omega(\mathrm{Pow}(C)) : C \text{ cyclic subgroup of } G\}.$$

So the problem is to find the clique number of the power graph of the cyclic group of order n.

Define the number-theoretic function f by $f(n) = \omega(\mathrm{Pow}(C_n))$, where C_n is the cyclic group of order n.

Theorem

The function f is given by the recurrence

- $f(1) = 1$;
- *for* $n > 1$,

$$f(n) = \phi(n) + f(n/p),$$

  where ϕ is Euler's totient function and p is the smallest prime divisor of n.

Proof.

The $\phi(n)$ generators of C_n are dominating vertices and so lie in every maximal clique. Show that the remainder of the clique is contained in a proper subgroup, and the best we can do is to take the largest proper subgroup. □

A bound and a constant

From this theorem, it is relatively easy to prove that $f(n) \leq 3\phi(n)$.
In fact,

$$\limsup(f(n)/\phi(n)) = 2.6481017597\ldots$$

The analytic formula for the constant is

$$\sum_{k\geq 0}\prod_{i=1}^{k}\frac{1}{p_i - 1},$$

where $p_1, p_2, \ldots$ are the primes in order.

Problem

Find all limit points of the ratio $f(n)/\phi(n)$. (The number above is the largest.)

All finite groups

We saw that $\omega(\mathrm{Pow}(G)) = \max\{f(m)\}$, where f is the function defined earlier and m runs over all orders of elements of G. In fact it is easy to see that, if $a \mid b$, then $f(a) \leq f(b)$; so it is enough to maximize over orders which are maximal in the divisibility order.

We note in passing that the maximal cliques in the enhanced power graph are maximal cyclic subgroups, so $\omega(\mathrm{EPow}(G)) = \max\{m\}$, where m runs over the same set.

If it were true that f were monotonic, then we would have $\omega(\mathrm{Pow}(G)) = f(\omega(\mathrm{EPow}(G))$, but this is false. For the group $G = \mathrm{PGL}(2, 11)$, the maximal orders of elements (in the divisibility ordering) are 10, 11 and 12; and $f(10) = f(12) = 9$, but $f(11) = 11$; so $\omega(\mathrm{EPow}(G)) = 12$ and $\omega(\mathrm{Pow}(G)) = 11$.

A bound

In any case, we have $\omega(\mathrm{Pow}(G)) \geq f(\omega(\mathrm{EPow}(G))$. Also $f(n) \geq \phi(n)$, and $\phi(n) \geq \mathrm{e}^{-\gamma} n / \log\log n$, where γ is the Euler–Mascheroni constant. Together these show that, if $\omega(\mathrm{Pow}(G)) = m$, then $\omega(\mathrm{EPow}(G)) \leq cm \log\log m$.

It is known that the inequality for $\phi(n)$ is essentially sharp, so this bound cannot be improved.

These results are in the survey with Ajay Kumar, Lavanya Selvaganesh and T. Tamizh Chelvam.

An observation and a problem

Theorem

A finite group G satisfies $\omega(\mathrm{Pow}(G)) = \omega(\mathrm{EPow}(G))$ if and only if the maximum order of an element of G is a prime power.

I do not have a simple description of this class of groups. However, this observation leads to some very general problems:

Question

Let A and A' be two graph types in the heirarchy, and p a monotonic graph parameter. Describe the finite groups G which satisfy $p(\mathrm{A}(G)) = p(\mathrm{A}'(G))$.

Results on this would extend known results on the groups for whith $\mathrm{A}(G) = \mathrm{A}'(G)$.

Matching number of power graphs

The matching number of a graph is the largest size of a set of pairwise disjoint edges.
V. V. Swathi talked about the matching number of power graphs in RDGG in June. Subsequently, she and I have been working on this.
If G has odd order, then $\mathrm{Pow}(G)$ has a matching covering all but one vertex – just match each element with its inverse. So assume that $|G|$ is even.
A tool for studying matching number is the "alternating chain argument". We used such an argument to show the following:

Theorem

Let G be a finite group, and let $S = \{g \in G : g^2 = 1\}$ be the set consisting of the identity and the involutions in G. Then there is a matching of maximum size in $\mathrm{Pow}(G)$ in which the set of unmatched vertices is contained in S.

Groups with unique involution

Corollary

Let G be a finite group with a unique involution. Then $\mathrm{Pow}(G)$ *has a perfect matching (one whose edges cover all vertices).*

Proof.

Take a matching as in the theorem. If there are unmatched vertices, they must be the identity and the involution; but these are joined, and we may add that edge to the matching. □

The converse is false.

Groups with a unique involution are essentially known, but the classification is not as well known as it should be. I will sketch how it goes. This is based on a paper I wrote with Laszlo Babai in 2000. (The classification question was asked by Coxeter, who observed that binary polyhedral groups have unique involutions.)

Theorem

- *Let G be a group with a unique involution z. Then $G/\langle z\rangle$ has cyclic or dihedral Sylow 2-subgroups.*
- *Conversely, let H be a group with cyclic or dihedral Sylow 2-subgroup. Then there is a group G, unique up to isomorphism, having a unique involution z, such that $G/\langle z\rangle \cong H$.*

Proof.

The first part is straightforward, since a Sylow 2-subgroup of G has a unique involution, and such groups were classified by Burnside: they are cyclic or generalized quaternion, and the quotient by the involution is cyclic or dihedral.
The second part depends on a cohomological argument suggested by George Glauberman; I refer to my paper with Babai. $\square$

Theorem

Let G be a group with cyclic or dihedral Sylow 2-subgroups. Let $O(G)$ be the largest normal subgroup of G of odd order. Then $G/O(G)$ is isomorphic to one of the following:

- *a subgroup of* $\mathrm{P\Gamma L}(2, q)$ *containing* $\mathrm{PSL}(2, q)$*, where q is an odd prime power;*
- *the alternating group A_7;*
- *a Sylow 2-subgroup of G.*

Proof.

This follows from Burnside's transfer theorem (in the cyclic case, when only the last conclusion is possible) and the Gorenstein–Walter theorem (in the dihedral case). □

Abelian groups

Theorem

Let G be a finite abelian group of even order. Let $I(G)$ be the set of involutions in G, and $O(G)$ the subgroup consisting of the elements of odd order in G.

- *If $|O(G)| \geq |I(G)|$, then $\mathrm{Pow}(G)$ has a perfect matching.*
- *Otherwise, the number of unmatched vertices in a matching of maximum size is $|I(G)| - |O(G)|$.*
- *The same statements hold for $\mathrm{EPow}(G)$.*

The first statement is false for non-abelian groups. The symmetric group S_3 has three involutions and three elements of odd order, but a maximum matching has only two edges, leaving two vertices uncovered.
However, in connection with the general problem posed earlier, we do not know a group G for which the matching numbers of $\mathrm{Pow}(G)$ and $\mathrm{EPow}(G)$ are unequal. Are they always equal??

A bigger playground

As a result of Lavanya Selvaganesh's talk, I have been able to enlarge considerably the class of potentially interesting graphs defined on groups. The hierarchy I lectured about is one-dimensional, but now it has grown into a second dimension.

Let A be a type of graph defined on groups (such as power graph or commuting graph), and B an equivalence relation on a group (such as conjugacy or "same order"). Let $[g]_B$ denote the B-equivalence class containing g.

I define the B superA graph of a group G as follows. The vertex set is the set of all elements of G. We join g to h if there are elements $g' \in [g]_B$ and $h' \in [h]_B$ such that $\{g', h'\}$ is an edge of the A graph on G.

Thus Lavanya's superpower graph is now called the order superpower graph, to distinguish it from the conjugacy superpower graph.

A two-dimensional hierarchy

Now as the type A of graph moves up through the hierarchy, the edge set of the A graph on G increases, and so the edge set of the B superA graph also increases.

But also, as the equivalence relation B becomes successively coarser, the edge set of the B superA graph also increases.

If B is the relation of equality, then the B superA graph of G is just the A graph. If G is abelian, then conjugacy coincides with equality, so the conjugacy superA graph is just the A graph. This is not the case in general.

Can we prove anything about these graphs? As a "proof of concept", I present a theorem about the groups for which one of these graphs is complete. In the table following, condition $(*)$ means that G contains an element whose order is equal to the exponent of G.

Theorem

The following table describes groups whose power graph, enhanced power graph, commuting graph, or their conjugacy or order supergraph is complete.

	power graph	enhanced power graph	commuting graph
basic	cyclic p-group	cyclic	abelian
conjugacy	cyclic p-group	cyclic	abelian
order	p-group	$(*)$	$(*)$

A sample proof

I will prove that the conjugacy supercommuting graph of G is complete if and only if G is abelian. From this it follows that, if either the conjugacy superpower graph or the conjugacy superenhanced power graph are complete, then G is abelian; so the second row of the table coincides with the first.

The "if" direction is trivial, so suppose that G is a group whose conjugacy supercommuting graph is complete. This means that, for any two elements $g, h \in G$, there is a conjugate of h with commutes with g.

An old result of Jordan says that if H is a proper subgroup of a group G, then there is a conjugacy class in G disjoint from H. Taking $H = C_G(g)$ (the centralizer of g in G), we see that, if $H \neq G$, there is a conjugacy class none of whose elements commutes with g, a contradiction. So $C_G(g) = G$, whence $g \in Z(G)$. But this should hold for all $g \in G$; so G is abelian.

Are zero-divisor graphs universal?

In my lectures I said something about which finite graphs can be embedded as induced subgraphs in various interesting graphs on groups.

Let us call a class of graphs universal if every finite graph is embeddable in some graph in the class.

I mentioned that the enhanced power graphs, the deep commuting graphs, the commuting graphs, and the generating graphs of finite groups are all universal. (Power graphs are not, since they are comparability graphs of partial orders, and hence so are all their induced subgraphs.)

After the RDGG talk by T. Tamizh Chelvam, the question was raised of whether zero-divisor graphs of rings are universal. I will report on this. By convention, in what follows, all rings are commutative rings with identity. The vertices of the zero-divisor graph are the zero-divisors, and a and b are joined if $ab = 0$.

General finite rings

Theorem

The zero-divisor graphs of finite commutative rings with identity are universal.

Proof.

A very simple proof of this (using only Boolean rings) was given by G. ArunKumar. We can represent any graph as an intersection graph (the vertices are subsets of a set, joined if their intersection is non-zero). So an intersection graph representation of the complement of a given graph embeds it in the zero-divisor graph of a Boolean ring (whose elements are all subsets of a set, with symmetric difference as sum and intersection as product). □

Local rings

A local ring is a ring with a unique maximal ideal. Any finite ring is isomorphic to a direct sum of local rings. Are the zero-divisor graphs of finite local rings universal?

Theorem

If R is a finite local ring whose maximal ideal is principal (generated by a single element), then its zero-divisor graph is a threshold graph.

Here a threshold graph is one whose vertices have weights, and two vertices are joined if the sum of their weights exceeds some threshold.
However, T. Kavaskar pointed out that this is not true in general. We have characterized rings whose zero-divisor graphs are threshold, and proved a universality result:

Theorem

Every finite graph is isomorphic to an induced subgraph of the zero-divisor graph of some finite commutative local ring.

… for being part of this research discussion group.

Please keep in touch, and see if we can prove more great results about graphs and groups!
As T. S. Eliot said, in a commentary on the *Bhagavad-Gita* in his poem *The Dry Salvages*,

Not fare well,
But fare forward, voyagers.

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