A hierarchy of graphs defined on groups

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So many new results were proved as a result of the research discussion that my survey is now out of date! I would like to thank Vijay and Aparna for this wonderful opportunity. I have spent so long on-line in Kochi that I now consider myself a virtual South Indian.

Graphs on groups

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The prototype for the graphs I will consider is the commuting graph of a finite group G. The vertex set is G, and vertices x and y are joined if and only if $xy = yx$. This graph encodes a remarkable amount of information about the group. For a simple example, the set of dominating vertices (joined to all others) is the centre, and the closed neighbourhood of x is the centralizer of x.

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In an important paper in 1955, the first step towards the Classification of Finite Simple Groups, Brauer and Fowler first used the commuting graph (without actually defining it!) A sample result: if G has more than one conjugacy class of involutions, then any two involutions have distance at most 3 in the commuting graph (with the identity omitted).

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- As in the case of Brauer and Fowler, we can use the graph to understand the group better.
- The graph may be interesting in its own right, or possibly useful as a network; we want to calculate various parameters of it.

I am going to give an example where a couple of interesting graphs can be produced in this way. This involves another graph defined on a group G, the power graph of G: we join x and y if one is a power of the other. This is example is taken from a recent survey of power graphs of groups, by Ajay Kumar, Lavanya Selvaganesh, T. Tamizh Chelvam, and me, published earlier this year.

The power graph of M_{11}

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We perform two further reductions. First, we call two elements closed twins if they have the same closed neighbourhood. We can shrink each closed twin class to a single vertex, giving a graph with 1540 vertives. Now analogously two elements are open twins if they have the same open neighbourhood; shrinking these equivalence classes to single vertices gives a graph with 1210 vertices. No further twin reduction is possible. Let Γ be the resulting graph.

The automorphism group of Γ is just the Mathieu group M_{11}. It acts with four orbits $O_1, \ldots, O_4$, with cardinalities 165, 165, 220 and 660. The matrix whose (i,j) entry is the number of edges from a fixed vertex of O_i into O_j is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

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We get another interesting bipartite graph with vertex set $O_2 \cup O_3$, two vertices joined if they lie in different orbits but have a common neighbour in O_4. This graph also has automorphism group M_{11}; it has bipartite blocks with sizes 165 and 220; it is semiregular with valencies 4 and 3, and has diameter and girth equal to 10.

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- The commuting graph $\mathrm{Com}(G)$: $xy = yx$; equivalently, $\langle x, y \rangle$ is abelian.
- The non-generating graph $\mathrm{NGen}(G)$: $\langle x, y \rangle \neq G$.

Other graphs include the deep commuting graph, nilpotence graph, solvability graph, and Engel graph.

The hierarchy

The power graph, enhanced power graph, and commuting graph form a hierarchy: that is, the edge set of each is a subset of that of the next, so that each graph is a spanning subgraph of the next. Moreover, unless G is a 2-generated abelian group, the non-generating graph lies above of the hierarchy. We can put the null graph at the bottom and the complete graph at the top.

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- The power graph of G is null if and only if G is the trivial group. (The identity is joined to all other elements.)
- The enhanced power graph of G is equal to the power graph if and only if G has no subgroup $C_p \times C_q$, where p and q are distinct primes; equivalently, every element of G has prime power order. Such groups are sometimes called EPPO groups. The question of classifying them was first raised by Higman in 1957. In 1963, Suzuki determined the simple groups with this property; the complete classification has been concluded recently.

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- The commuting graph of G is equal to the enhanced power graph if and only if G has no subgroup $C_p \times C_p$ with p prime. Equivalently, the Sylow subgroups of G are cyclic or possibly (for $p = 2$) generalized quaternion. Such groups are also classified.

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There are various ways we could try to extend these results. For example, suppose that p is a monotonic graph parameter (that is, adding edges cannot decrease its value). We could ask, for which groups do two graphs in the hierarchy have the same value of p?

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Here is a simple example.

Theorem

The clique numbers of the power graph and enhanced power graph of a group G are equal if and only if the largest order of an element of G is a prime power.

Here is a less trivial example, which I proved with V V Swathi and M S Sunitha.

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Theorem

For any finite group G, the power graph and enhanced power graph of G have the same matching number.

This holds even though we do not have a formula for the matching number! The proof involves taking a maximum matching in the enhanced power graph, and showing that edges not in the power graph can be replaced by edges in the power graph to get a matching of the same size.

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Plenty of open problems!

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Now the power graph is the comparability graph of the directed power graph, which is a reflexive relation (if we add loops) and is also transitive, that is, a partial preorder. It is easy to show that it can be refined to a partial order without changing the comparability graph. Hence:

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Theorem

The power graph of a finite group is the comparability graph of a partial order, and hence is perfect.

Universality

Not every graph is embeddable as induced subgraph in the power graph of a finite group, since any induced subgraph must be a comparability graph (and so perfect). But this is the only restriction. Moreover, for the other graph types in the hierarchy, there is no restriction:

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- *Let Γ be the comparability graph of a finite partial order. Then there is a finite group G such that Γ is embeddable as induced subgraph in the power graph of G.*
- *For any other graph type in the hierarchy, for any finite graph Γ, there is a finite group G such that Γ is embedded in the graph of that type defined on G.*

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In particular, I would like to know which 3-colourings can be embedded so that red edges belong to the power graph, green are in the enhanced power graph but not the power graph, and blue not in the enhanced power graph.
It is necessary that the red edges form the comparability graph of a partial order; also, if (a, b, c, d) is an induced path in the red graph, then at least one of $\{a, c\}$ and $\{b, d\}$ must be green. Are these conditions sufficient?

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- *The vertices joined to all others in the commuting graph of G form the centre of G.*

Connectedness

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I will give a couple of the most spectacular recent results on connectedness of some of these graphs. For graphs in the hierarchy, the reduced graph means the graph obtained by deleting the vertices joined to all others. (However, the generating graph is the complement of the non-generating graph, so it is more natural to delete the isolated vertices.)

The first theorem is due to Michael Giudici, Luke Morgan and Chris Parker, and the second to Saul Freedman, and the third to Tim Burness, Robert Guralnick and Scott Harper.

The commuting graph

Theorem

- *There is no upper bound for the diameter of the reduced commuting graph of a finite group; for any given d there is a 2-group whose reduced commuting graph is connected with diameter greater than d.*

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- *Suppose that the finite group G has trivial centre. Then every connected component of its reduced commuting graph has diameter at most 10.*

The non-generating graph

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It is possible that 5 can be reduced to 4 here. This would be best possible since there are finite simple graphs whose reduced groups whose reduced non-generating graph has diameter 4.

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In particular, if G is non-abelian simple, then deleting the identity from the generating graph gives a graph of diameter 2.

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The first statement is easy since, if a set of elements in a group G commute pairwise, then they generate an abelian subgroup of G.
For the second, an analogous statement holds: if a set of elements pairwise generate cyclic groups, then all together they generate a cyclic group. This is a little less straightforward to prove . . .

Clique number of power graph

Using the principle on the last slide, we see that a clique in the power graph of G is contained in a cyclic subgroup of G. So the clique number of G is equal to the maximum clique number of a cyclic subgroup of G. Since the power graph is perfect, this is also the chromatic number.

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Theorem

The function f is given by the number-theoretic recurrence

- $f(1) = 1$;
- *for $n > 1$, $f(n) = \phi(n) + f(n/p)$, where ϕ is Euler's totient function and p is the smallest prime divisor of n.*

The inductive step is proved by noting that the $\phi(n)$ generators of C_n are joined to all other vertices, so lie in every maximal clique; then show that the remaining vertices lie in a proper subgroup, and choosing the largest subgroup gives the largest value.

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Proposition

$$\limsup f(n)/\phi(n) = 2.6481017597\ldots,$$

where the constant on the right is given by

$$\sum_{k\geq 0} \prod_{i=1}^{k} \frac{1}{p_i - 1},$$

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The set $\{f(n)/\phi(n) : n \in \mathbb{N}\}$ has other limit points: *can we find them all?*

The function f is monotonic with respect to divisibility, but not with the usual order; so it is not the case the $\omega(\mathrm{Pow}(G)) = f(\omega(\mathrm{EPow}(G)))$. For example, in the group $G = \mathrm{PSL}(2, 11)$, the maximal orders with respect to divisibility are 10, 11 and 12, so $\omega(\mathrm{EPow}(G)) = 12$; but $f(10) = 9 = f(12)$ but $f(11) = 11$, so $\omega(\mathrm{Pow}(G)) = 11$.

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- A nilpotent group whose power graph is a cograph is either a group of prime power order or C_{pq} where p and q are distinct primes.
- A non-abelian simple group whose power graph is a cograph is either $\mathrm{PSL}(2, q)$ or $\mathrm{Sz}(q)$ for certain prime powers q, or $\mathrm{PSL}(3, 4)$.

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