A quick tour of algebraic graph theory

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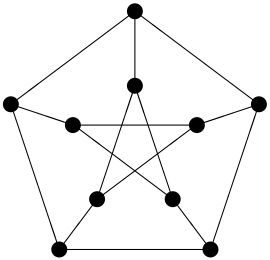
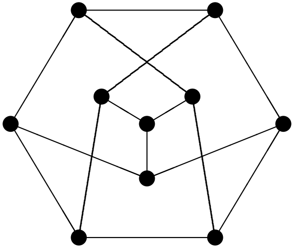
Since Stephan Wagner will be talking about spectral graph theory on Thursday I will spend more time on the second part.

Two graphs





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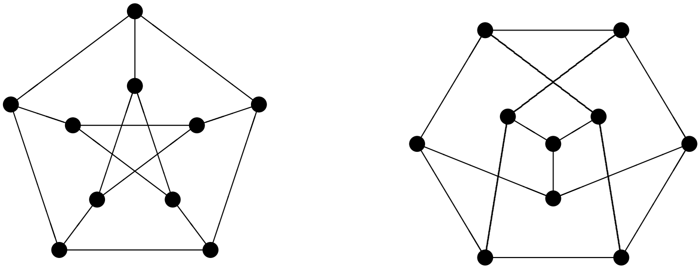

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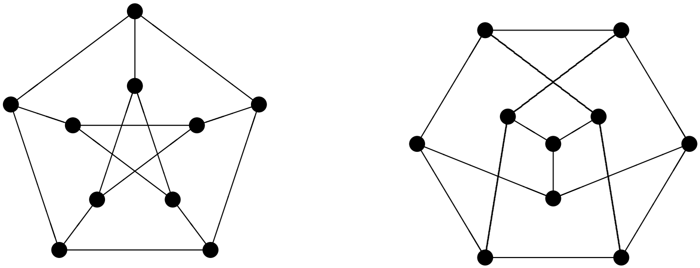

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Try it. You will fairly quickly construct an isomorphism.

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However, for graphs of manageable size, practical algorithms such as Brendan McKay's `nauty` out-perform Babai's algorithm.

In matrix terms . . .

Let Γ be a graph with n vertices. Number the vertices from 1 to n, and define the adjacency matrix $A = A(\Gamma)$ to be the $n \times n$ matrix with (i, j) entry 1 if the ith and jth vertices are adjacent, and 0 if not.

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Then A is a real symmetric matrix, so there is an orthogonal matrix P such that $PAP^\top$ is diagonal; the diagonal entries are the eigenvalues of A. Said otherwise, $\mathbb{R}^n$ is an orthogonal direct sum of eigenspaces of A.

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Thus, isomorphic graphs have the same eigenvalues and multiplicities.

Generalized line graphs

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A cocktail party graph $\mathrm{CP}(m)$ is the graph with $2m$ vertices $a_1, \ldots, a_m, b_1, \ldots, b_m$ in which a_i is joined to every vertex except b_i (and the same with a and b reversed).

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Given a labelling l of the vertices of Γ with non-negative integers, the corresponding generalized line graph is the union of $L(\Gamma)$ with cocktail party graphs $\mathrm{CP}(l(v))$ for all vertices v of Γ, where an edge $\{v, w\}$ is joined to the cocktail party graphs corresponding to v and w.

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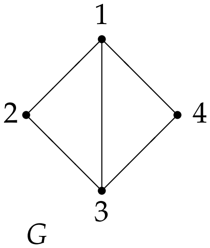
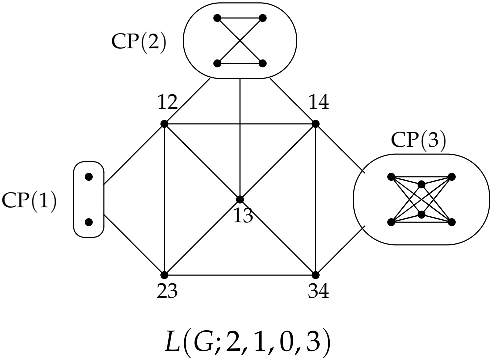
The next slide gives an example.

A generalized line graph

The figure shows a graph Γ and the generalized line graph $L(\Gamma;(2,1,0,3))$.





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The theorem is proved using the concept of root systems from the theory of Lie algebras (though these arise in many other areas of mathematics). The Petersen graph has smallest eigenvalue -2 and is one of the finite list of exceptions. In fact, the exceptional graphs are all represened by subsets of the exceptional root system E_8 with all products non-negative, two vertices joined if their inner product is positive.

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The complete list has not been computed, but we know all the regular graphs with smallest eigenvalue -2 which are not line graphs. Unsurprisingly, the Petersen graph is one of these.

Applications

The theorem has various applications in graph theory. For example, it is not hard to show that, if a generalised line graph is regular, then it must be a line graph or a cocktail party graph; so a regular graph with least eigenvalue -2 must be of one of these types.

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There are applications further afield. Peter Sarnak, who was this year's London Mathematical Society Hardy Lecturer, told us about an application to an engineering problem involving the design of waveguides.

Cospectral graphs

The spectrum of the adjacency matrix does not determine the graph up to isomorphism. (If graphs with adjacency matrices A and B are cospectral then $B = PAP^\top$ for some orthogonal matrix P; to be isomorphic we require P to be a permutation matrix.) This is why linear algebra doesn't solve the graph isomorphism problem.

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Indeed there can be more than exponentially many graphs on n vertices with the same spectrum, something which we now explore.

Strongly regular graphs

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Two strongly regular graphs are cospectral if and only if they have the same parameters.

Indeed the eigenvalues and their multiplicities can be calculated from the parameters. The fact that the multiplicities are non-negative integers puts constraints on the parameters, which are necessary conditions for the existence of strongly regular graphs.

Latin squares

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Now the number of non-isomorphic Latin squares is very roughly $(n/c)^{n^2}$ for some constant c. This is asymptotically bigger than $\exp(n^2)$, giving more than exponentially many cospectral graphs. Moreover, almost all of them have trivial automorphism group.

Steiner triple systems

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As before, for admissible values of n, there are so many of them that we again obtain more than exponentially many cospectral graphs, almost all of which have trivial automorphism groups.

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Theorem

Let Γ be a strongly regular graph with smallest eigenvalue -3. Then either

- Γ *is a complete multipartite graph with parts of size* 3*;*
- Γ *is a Latin square graph;*
- Γ *is a Steiner graph;*
- Γ *belongs to a finite list of exceptions.*

Regularity and symmetry

Being strongly regular is a very strong regularity condition on a graph, but as we have seen, it does not imply the existence of any symmetry. Can we strengthen the condition so as to obtain symmetry from regularity?

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A graph Γ is t-homogeneous if any isomorphism between induced subgraphs of Γ on at most t vertices extends to an automorphism of Γ. It is homogeneous if it is t-homogeneous for all $t \leq n$, where n is the number of vertices. This is a very strong symmetry condition.

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Using the Classification of Finite Simple Groups, it is possible to write down a list of all the 2-homogeneous finite graphs.

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- *the line graph of $K_{3,3}$.*

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Let G be a finite group. Let $a_n(G)$ be the number of n-vertex graphs Γ for which $G \leq \mathrm{Aut}(\Gamma)$, and $b_n(G)$ the number for which equality holds. Then $b_n(G)/a_n(G)$ tends to a limit as $n \to \infty$.

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The limit is not always one. Indeed, for nilpotent groups, values of the limit are dense in $[0, 1]$.

Vertex-transitive graphs

The graph Γ is vertex-transitive if, for any two vertices, there is an automorphism of Γ mapping one to the other. More generally, if G is a subgroup of $\mathrm{Aut}(\Gamma)$, then Γ is G-vertex-transitive if the automorphism in the definition can be chosen to lie in G.

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To examine vertex-transitive graphs further, we need to look at permutation groups.

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We say that G acts transitively on Ω if, for all $\alpha, \beta \in \Omega$, there exists $g \in G$ with $\alpha g = \beta$.

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It is in fact a subgroup of Ω. Stabilisers of different points in a transitive group are conjugate subgroups.
We say that G acts regularly on Ω if it acts transitively on Ω and $G_\alpha = \{1\}$ for some (and hence all) $\alpha \in \Omega$.

Orbitals and orbital graphs

Let G be transitive on Ω. An orbital of G is an orbit of G on $\Omega \times \Omega$, the set of ordered pairs of elements of Ω. Thus there is a unique diagonal orbital $\{(\alpha, \alpha) : \alpha \in \Omega\}$. G acts 2-transitively if there is a unique non-diagonal orbital (that is, any two distinct elements can be mapped to any other such pair by an element of G).

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Each orbital graph admits G as an arc-transitive group of automorphisms. If O is not self-paired and we define the graph with edge set the union of O and the paired orbital, then we obtain an undirected orbital graph, on which G acts edge-transitively.

All G-invariant graphs

Proposition

Let G be a transitive permutation group on Ω. Then any graph on the vertex set Ω which is G-invariant has edge set the union of some self-paired orbitals amd some pairs of paired orbitals for G.

A similar result describes all the G-invariant directed graphs: the edge sets are arbitrary unions of orbitals.

Cayley graphs

If G acts regularly on Ω, then we can identify Ω with G, where an arbitrary element α of Ω is identified with the identity, and then $\beta = \alpha g$ is identified with $g \in G$.

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With this identification, the action of G on itself is by right multiplication. The orbitals are the sets $O_s = \{(x, sx) : x \in G\}$ for each $s \in G$; the orbital O_s is paired with $O_{s^{-1}}$. Thus a G-invariant undirected graph on G has edge set $\{\{x, sx\} : x \in G, s \in S\}$, where S is an inverse-closed subset of $G \setminus \{1\}$. Such a graph is called a Cayley graph for G.

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In particular, Cayley graphs form an important subclass of vertex-transitive graphs.

Vertex-transitive graphs and Cayley graphs

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For example, consider the Petersen graph. Its automorphism group is isomorphic to the symmetric group S_5; the only subgroup of order 10 in S_5 is the dihedral group, which contains involutions. But the vertices of the Petersen graph are identified with 2-sets from the set on which S_5 acts; and an involution fixes the 2-sets corresponding to its cycles of length 2.

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Investigations by Brendan McKay, Cheryl Praeger and others seem to show that "most" vertex-transitive graphs are Cayley graphs, but as yet this is unproven. However, Gerd Sabidussi showed that every vertex-transitive graph has a cover (in a suitable sense) which is a Cayley graph.

A warning

In the literature you will meet the concept of a normal Cayley graph. Unfortunately, the term has two incompatible meanings, both of which are very interesting.

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I'd like to change the terminology. Maybe we could call the first type inner-automorphic since it is preserved by inner automorphisms of G?

Graphs on groups

I will finish with a topic which has recently seen a lot of interest: this concerns graphs whose vertex set is a group, but unlike (most) Cayley graphs it reflects the structure of the group in some way, and it is invariant under all automorphisms of the group.

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For a survey with more details, see my forthcoming survey in the *International Journal of Group Theory*.

A hierarchy

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- The complete graph.

Directed power graph

There is also a directed power graph, with an arc $x \to y$ if y is a power of x. This relation is reflexive (if we add loops at each vertex) and transitive, that is, a partial preorder, and the power graph is its comparability graph. Using this we can show that the power graph is the comparability graph of a partial order, and hence (by Dilworth's Theorem) it is perfect (that is, any induced subgraph has clique number equal to chromatic number).

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We do not have a good characterization of pairs of groups for which these conditions hold.

Isoclinism

The notion of isoclinism of groups was introducd by Philip Hall. Roughly it says that the commutation structure of two groups is the same.

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Commutation in a group G can be regarded as a map $\gamma : G/Z(G) \times G/Z(G) \to G'$, where $Z(G)$ and G' are the centre and derived subgroup of G. We say that G_1 and G_2 (with commutation maps γ_1 and γ_2) are isoclinic if there are isomorphisms $\alpha : G_1/Z(G_1) \to G_2/Z(G_2)$ and $\beta : G_1' \to G_2'$ such that

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The converse is known in some cases: simple groups, abelian groups, extraspecial groups, … . Indeed, I know no examples where it fails.

Universality

We saw that the power graph is the comparability graph of a partial order. In terms of induced subgraphs, this is the only restriction; and there is no comparable restriction for the other graphs in the hierarchy:

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- *Let Γ be any finite graph. Then, for any of the other graph types in the hierarchy, there is a group G such that Γ is an induced subgraph of the graph of that type defined on the group G.*

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- *Let Γ be any finite graph. Then, for any of the other graph types in the hierarchy, there is a group G such that Γ is an induced subgraph of the graph of that type defined on the group G.*

Some questions remain. For example, what is the smallest group required to embed a given graph, or to embed all graphs of a given order?

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- The commuting graph is equal to the enhanced power graph if and only if G has no subgroup $C_p \times C_p$ for p prime. These are the groups whose Sylow subgroups are cyclic or generalized quaternion; again they have all been classified.

- For G not 2-generated abelian, the commuting graph is equal to the non-generating graph if and only if G is a minimal non-abelian group. These groups were all determined by Miller and Moreno in 1904.

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If two graphs in the hierarchy are not equal, we can ask about the graph whose edge set is the difference of their edge sets. In the extreme cases, the difference between the power graph and the null graph is the power graph, while the difference between the complete graph and the non-generating graph is the generating graph; both of these have been extensively studied.

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In the power graph of a group G, every edge (and hence every clique) is contained in a cyclic subgroup of G. So the clique number is the maximum clique number of the cyclic subgroups of G.
Define the function f by the rule that $f(n)$ is the clique number of the power graph of C_n. Then f is given by the recurrence

- $f(1) = 1$;
- $f(n) = \phi(n) + f(n/p)$, where ϕ is Euler's totient and p the smallest prime divisor of n.

Using this, one can show that $\phi(n) \leq f(n) \leq 3\phi(n)$. In fact,

$$\limsup f(n)/\phi(n) = 2.6481017597\ldots;$$

the constant on the right is

$$c = \sum_{k \geq 0} \prod_{i=1}^{k} \frac{1}{p_i - 1},$$

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where $p_1, p_2, \ldots$ are the primes in order.
This constant is a limit point of values of $f(n)/\phi(n)$. Can all the limit points be described?

The enhanced power graph

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So, if p is a monotone graph parameter, then $p(\mathrm{Pow}(G)) \leq p(\mathrm{EPow}(G))$. Asking when equality holds is a generalisation of the problem of determining the EPPO groups.

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Here is one parameter for which there is a surprising answer:

Theorem

For any finite group G, the matching numbers of the power graph and the enhanced power graph of G are equal.

However, we do not have a general formula for the matching number!

The commuting graph

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Theorem

- *There is no upper bound for the diameter of the reduced commuting graph of a finite group; for any given d there is a 2-group whose reduced commuting graph is connected with diameter greater than d.*

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The following striking result is due to Michael Giudici, Luke Morgan and Chris Parker:

Theorem

- *There is no upper bound for the diameter of the reduced commuting graph of a finite group; for any given d there is a 2-group whose reduced commuting graph is connected with diameter greater than d.*
- *Suppose that the finite group G has trivial centre. Then every connected component of its reduced commuting graph has diameter at most 10.*

The generating graph

The following theorem of Tim Burness, Bob Guralnick and Scott Harper settles an old conjecture. Let Γ be the generating graph of G with the identity vertex removed.

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In particular, these conditions hold if G is a non-abelian finite simple group.

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… for your attention.