Triangle-free strongly regular graphs: A themed series of exercises

Peter J. Cameron
University of St Andrews

Johannesburg Graph Theory School
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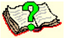
I am going to tell you a story, which involves seven beautiful graphs. On the way we will meet a number of other topics in discrete mathematics, from the outer automorphism of the symmetric group S_6 to the Steiner system on 22 points.

At several points along the way, I will not give proofs: your job is to provide proofs of the assertions I make. I will use the tag  to indicate things you could try to prove. Some are easy, some are hard. I have produced a separate file containing these problems and solutions, available on request.

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We will be talking about strongly regular graphs with no triangles (that is, with $\lambda = 0$). For any n, the complete bipartite graph $K_{n,n}$ is strongly regular, with parameters $(2n, n, 0, n)$.

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[Why not the 4-cycle C_4? Because it is $K_{2,2}$.]

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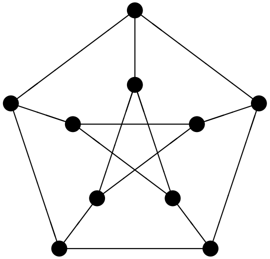

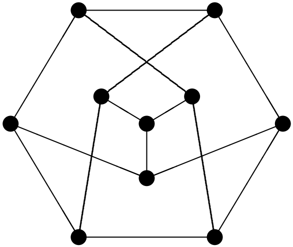

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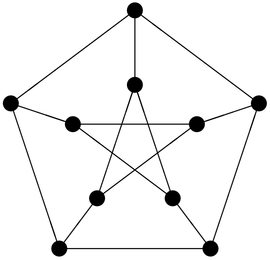

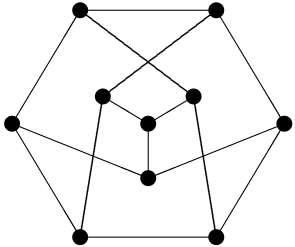

It can also be described as follows. The vertices are the 2-element subsets of a set of size 5, two vertices joined if the subsets are disjoint. These descriptions all give the same graph.

To discuss the Clebsch graph, I will first introduce the n-cube Q_n. The vertices are all the subsets of the set $\{1, 2, \ldots, n\}$; two vertices A and B are joined if their symmetric difference has size 1, that is, if either $B = A \cup \{x\}$ with $x \notin A$, or $A = B \cup \{x\}$ with $x \notin B$. It is a regular graph with valency n on 2^n vertices; it has diameter n, and is antipodal: that is, for every vertex, there is a unique vertex at distance n from it (the complementary subset).

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- Take the 5-cube and identify antipodal vertices.
- Take the 4-cube and add edges joining antipodal pairs of vertices.

These two graphs are isomorphic. This is the Clebsch graph. It is strongly regular with parameters $(16, 5, 0, 2)$. Moreover, the set of vertices at distance 2 from a given vertex induces the Petersen graph. You should prove all this. 

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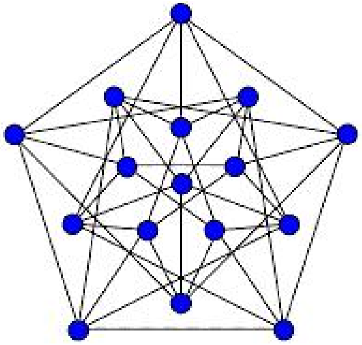

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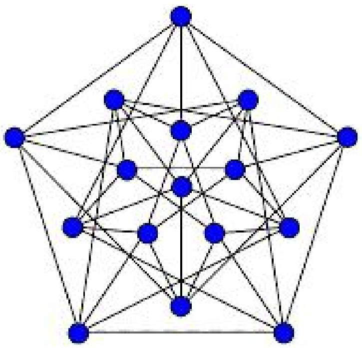

If you delete the central vertex and its five neighbours, you will recognise what is left as our picture of the Petersen graph with fivefold symmetry.

Spectral theory

Let Γ be a strongly regular graph with parameters (n, k, λ, μ). Let A be its adjacency matrix, J the all-1 matrix of order n, and I the all-1 identity matrix. Note that $B = J - I - A$ is the adjacency matrix of the complement of Γ.

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So A has an eigenvalue k corresponding to the all-1 vector $\mathbf{j}$; it preserves the space $\mathbf{j}^\perp$ of real vectors with coordinate sum 0, and on this space satisfies the quadratic equation $A^2 = (k - \mu)I + (\lambda - \mu)A$. Hence it has just two eigenvalues on $\mathbf{j}^\perp$.

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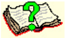
Conversely, if Γ is a graph whose adjacency matrix has $\mathbf{j}$ as an eigenvector and has just two eigenvalues on $\mathbf{j}^\perp$, then Γ is strongly regular. 

Uniqueness

It can be shown that the Petersen and Clebsch graphs are the unique strongly regular graphs with their parameters. 

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We are going to use this to look at a problem due to Allan Schwenk.

Partition into Petersens

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In other words, if we can find two disjoint Clebsch graphs, then what remains is also a Clebsch graph.

Finding a partition

Let F be a finite field of order 16. Its multiplicative group has order 15, and so the field contains five 5th roots of unity. Its cosets are the sets of 5th roots of the three non-zero elements in the subfield of order 4. Let A, B, C be these three sets.

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Now consider the Cayley graphs of the additive group of F with A, B and C as connection sets. Now one can show that no set of three or four of the 5th roots of unity can sum to zero; so the first Cayley graph has no cycles of length 3, and none of length 4 except for $(x, x+a, x+a+a', x+a', x)$. So the first Cayley graph is strongly regular with parameters $(16, 5, 0, 2)$. Hence it is a Clebsch graph.

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Multiplication by a cube root of unity permutes A, B, C cyclically, so all three of the Cayley graphs is isomorphic to the Clebsch graph.

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What about three colours? It is not hard to show that, in K_{17}, if the edges are coloured with three colours, then there is a monochromatic triangle. Seventeen is best possible. For in the partition into three Clebsch graphs just constructed, if we colour them with three different colours, there is no monochromatic triangle.

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This argument is due to Greenwood and Gleason.

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For $k = 2$ and $k = 3$, there are unique graphs, namely the first two of our magnificent seven: the 5-cycle and the Petersen graph. Hoffman and Singleton constructed and showed unique a Moore graph with valency 7; I will show you how this can be done. The existence of a Moore graph of valency 57 is still unknown.

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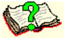
In the case $d=2$, such a graph is strongly regular with parameters $(n,k,0,1)$, with $n=k^2+1$. Analysis of their eigenvalues and their multiplicities shows that necessarily $k=2,3,7$ or 57. 
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For larger diameters, Bannai and Ito, and independently Damerell, showed that the only possible graph is the $2d+1$-cycle.

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- 15 one-factors (sets of three edges partitioning the vertex set);
- 6 one-factorizations (sets of five one-factors partitioning the edge set).

Let B be the set of 1-factorizations.

The outer automorphism of S_6

It is a remarkable fact, first established by Sylvester, that the symmetric group S_6 has an outer automorphism (one not induced by conjugation by a group element). In fact, no other symmetric group, finite or infinite, has an outer automorphism.
I will sketch Sylvester's construction.
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Two 1-factorizations share a unique common 1-factor, so 1-factors can be identified with edges on B. Similarly, edges on A can be identified with 1-factors on B, and points of A with 1-factorizations on B. So doing the process twice brings us back.

Now it is clear that any permutation π on A induces a permutation π^* on B; moreover $(\pi\sigma)^* = \pi^*\sigma^*$. so the map $\pi \mapsto \pi^*$ is an isomorphism from $S_6(A)$ to $S_6(B)$, in other words, induces an automorphism of S_6. It is not an inner automorphism, since the stabilizer of a 1-factorization fixes no point of A.

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Now we are going to use this set-up to construct a very interesting graph, which was named the Sylvester graph by Norman Biggs.

The Sylvester graph

As described, let A be a 6-set and B the set of six 1-factorizations on A. Construct a graph as follows:

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- The graph has valency 5.
- A vertex and its five neighbours have the property that one is in each row and one in each column of the square array $A \times B$.
- Conversely, if two vertices lie in different rows and different columns, then they lie at distance 1 or 2 in the graph. Vertices in the same row or in the same column are at distance 3.

A curious sidelight

What is special about the number 6? We have seen that the symmetric group of degree 6 has an outer automorphism, which can be used to construct various combinatorial objects. (We will soon see the Hoffman–Singleton graph, but others include the projective plane of order 4 and the Steiner system $S(5, 6, 12)$.)

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Recently in a paper with Rosemary Bailey, Leonard Soicher and Emlyn Williams, we showed how the Sylvester graph can be used to construct substitute designs which would be nearly as good as the (non-existent) square lattice designs.

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It is strongly regular with parameters $(50, 7, 0, 1)$, in other words, a Moore graph.

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The number of edges of the Hoffman–Singleton graph is one-seventh of the number of edges of K_{50}. Can K_{50} be partitioned into seven Hoffman–Singleton graphs?

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The authors assumed the set of graphs invariant under a non-trivial group. So if the partition exists, it has no non-trivial automorphisms.

Orthogonal Latin squares

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Now we start on the road to building the last three graphs. First, the strongly regular Latin square graphs can be generalised. Two Latin squares L_1 and L_2 of the same order are orthogonal if given a pair (a, b) of letters, there is a unique cell (i, j) such that the (i, j) entry of L_1 is a while that of L_2 is b. Here is an example:

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a	b	c
b	c	a
c	a	b

p	q	r
r	p	q
q	r	p

$\rightarrow$

ap	bq	cr
br	cp	aq
cq	ar	bp

A set of Latin squares is mutually orthogonal if any two of the squares are orthogonal.
The size of a set of mutually orthogonal Latin squares (or MOLS) of order n is at most $n - 1$.

Pseudo Latin square graphs

Given a set of $s-2$ mutually orthogonal Latin squares of order n, make a graph whose vertices are the n^2 cells, with two cells joined if they lie in the same row or the same column or have the same entry in one of the squares.

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Any strongly regular graph with these parameters is called a pseudo Latin square graph.

Bruck proved that if s is not too large (roughly the fourth root of n), then any pseudo Latin square graph actually arises from a set of MOLS.

Negative Latin square graphs

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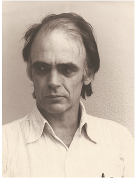
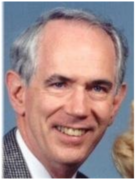
In his thesis, Mesner constructed a graph with these parameters.

Higman and Sims

On 3 September 1967, Donald Higman and Charles Sims were at a group theory conference in Oxford. Marshall Hall had just announced the construction of the simple group discovered by Zvonimir Janko, as a permutation group on 100 points (actually a subgroup of index 2 in the automorphism group of a pseudo-Latin square graph with parameters $(100, 36, 14, 12)$). At the conference dinner, Higman and Sims wondered whether there might be another sporadic simple group which was also a permutation group on 100 points. By the end of the evening they had found one.

R. D. Carmichael had constructed in 1931, and Ernst Witt proved unique in 1938, a configuration with 22 points and 77 blocks (we would call this a Steiner system $S(3, 6, 22)$) whose automorphism group contained the Mathieu group M_{22} as a subgroup of index 2. Higman and Sims built a graph from Witt's design. The vertex set consisted of the points and the blocks and one additional point $*$; the edges were given by three simple rules:

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Now they had to show that the graph looks the same from any point; this follows from standard properties and uniqueness of the design. It follows that its automorphism group is transitive, and contains a (new) simple group as a subgroup of index 2.

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See the paper by Misha Klin and Andrew Woldar in *Acta Universitatis Matthiae Belii*, series Mathematics, **25** (2017), 5–62, for further background and discussion: `http://actamath.savbb.sk`.

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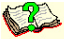
For a challenging exercise, prove that these recipes describe strongly regular graphs with the stated parameters. 

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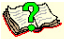
For a challenging exercise, prove that these recipes describe strongly regular graphs with the stated parameters. 

As a final note, the vertex set of the Mesner graph can be partitioned into two subsets of size 50 so that the induced subgraph on each subset is the Hoffman–Singleton graph.

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Theorem

The block graph of a quasi-symmetric 2*-design is strongly regular.*

For, if M is the incidence matrix of the design, then $MM^\top$ is a linear combination of I and J, while $M^\top M$ is a linear combination of I, J and the adjacency matrix of the graph; and these two matrices have the same eigenvalues with the same multiplicities apart from the eigenvalue 0.

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And with that I will conclude; thank you for your attention.