These are the problems highlighted in my problem-session talk on strongly regular graphs with no triangles. Try the problems yourself before looking at my solutions (which are not necessarily the only possible solutions).

Problem 1 Show that the complete bipartite graph $K_{n,n}$ is strongly regular, with parameters $(2n, n, 0, n)$.

Problem 2 Show that several descriptions of the Petersen graph are all isomorphic.

Problem 3 Show that the 5-cube with antipodal vertices identified and the 4-cube with extra edges joining antipodal vertices are both strongly regular with parameters $(16, 5, 0, 2)$ (and are isomorphic).

Problem 4 Show that if a graph has $\mathbf{j}$ as an eigenvector and just two eigenvalues on $\mathbf{j}^{\perp}$, then it is strongly regular.

Problem 5 Show that the Petersen and Clebsch graphs are the unique strongly regular graphs with their parameters.

Problem 6 Partition the edges of K_{10} into three graphs of which two are isomorphic to the Petersen graph.

Problem 7 Show that, if we have two disjoint Clebsch graphs in K_{16}, then the remaining edges form another Clebsch graph.

Problem 8 Show that the Cayley graph of the additive group of the field of order 16 with connection set a multiplicative coset of the fifth roots of unity is isomorphic to the Clebsch graph.

Problem 9 Prove that, if the edges of the complete graph on 17 vertices are 3-coloured, there must be a monochromatic triangle.

Problem 10 Show that a Moore graph of diameter 2 must have valency 2, 3, 7 or 57.

Problem 11 Show that there are just six 1-factorizations of K_6.

Problem 12 Show that two cells in the same row or column of the 6×6 array are at distance 3 in the Sylvester graph.

Problem 13 Verify that the Hoffman–Singleton graph as described is a Moore graph.

Problem 14 Show that there are at most $n-1$ mutually orthogonal Latin squares of order n.

Problem 15 Show that the graph associated with a set of $s-2$ MOLS is strongly regular with parameters $(n^2, s(n-1), s^2-3s+n, s(s-1))$.

Problem 16 Show that the Higman–Sims construction gives a strongly regular graph with parameters $(100, 22, 0, 6)$.

Problem 17 Verify that the last two graphs are strongly regular with the parameters asserted.

Problem 18 Show that the Cameron graph is strongly regular with the parameters asserted.

Solution 1 The complete bipartite graph has $2n$ vertices, partitioned into two sets A and B each of size n; edges join all points of A to all points of B.

Each vertex is joined to all n vertices in the other part of the partition.

There are no triangles, since the graph is bipartite.

If two vertices v and w are not joined, then the lie in the same part of the partition, say A (without loss); the common neighbours are all points of B.

Solution 2 We will show that a graph on 10 vertices with valency 3, diameter 2, and no cycles of length 3 or 4 is unique up to isomorphism. Then it is just a case of verifying these properties for all the given graphs.

Take an edge $\{a, b\}$. Now a has two further neighbours, say c_1 and c_2, and b has two further neighbours, say d_1 and d_2. All these vertices are distinct, since equality would create a triangle.

Since the diameter is 2, the vertices c_i and d_j have a neighbour e_{ij} for $i, j \in \{1, 2\}$ (and only one, or there would be a short cycle). Moreover, all these vertices are distinct. Now we have ten vertices and 13 of the 15 required edges; the e_{ij} only have valency 2 so far, so each must lie on one further edge. The edges must be $\{e_{11}, e_{22}\}$ and $\{e_{12}, e_{21}\}$, since any other choice would create a triangle (e.g. e_{11} and e_{12} are both joined to c_1).

Solution 3 The vertices of the 5-cube are all the subsets of $\{1, \ldots, 5\}$; two vertices are antipodal if the subsets form a partition of $\{1, \ldots, 5\}$. So the graph obtained by identifying antipodal vertices has vertex set all partitions of $\{1, \ldots, 5\}$ into (at most) two parts, of which there are 16; a the neighbours of a given vertex are all the partitions obtained by transferring a single point to the other part. (e.g. 12|345 is joined to 2|1345, 1|2345, 123|45, 124|35 and 125|34.) So there are 16 vertices, and the valency is 5. From this description we see there are no triangles. If two vertices are at distance 2, then two points have to be transferred between parts, and this can be done in either order, giving two paths joining them.

Now re-label the vertices with subsets of $\{1, \ldots, 4\}$ as follows: choose the part of the partition containing 5, and remove 5. This identifies the vertices with those of the 4-cube. Edges within the 4-cube are unchanged; but the result of transferring 5 to the other part of the partition replaces a subset by its complement (the antipodal vertex in the 4-cube). For example, 125|34 is represented by 12, and 12|345 by the complementary set 34.

So the two graphs are isomorphic.

Solution 4 Having $\mathbf{j}$ as an eigenvector means that $A\mathbf{j} = k\mathbf{j}$, so the graph is regular with valency k.

Now a matrix which preserves $\mathbf{j}$ and acts as the zero transformation on $\mathbf{j}^\perp$ is a multiple of the all-1 matrix J. Hence the adjacency matrix of our graph satisfies $(A - \alpha I)(A - \beta I) = \gamma J$, which when unpicked shows that A^2 is a linear combination of I, A and J; so the graph is strongly regular.

Solution 5 We already did the Petersen graph.

Let Γ be strongly regular with parameters $(16, 5, 0, 2)$. Take a vertex and call it ∞. Let A be the set of its five neighbours, and B the set of its ten non-neighbours. Then each vertex of B is joined to two vertices in A. Also the vertices in A are pairwise non-adjacent, so any two are joined to two common neoghbours, ∞ and a vertex in B. Thus the vertices of B are bijective with the set of 2-subsets of A (each vertex of B corresponding to its neighbours in A).

If two vertices of B are joined, then their labelling 2-sets must be disjoint (otherwise we have a triangle). But a vertex of B has three neighbours in B, and its label is disjoint from three 2-subsets of A. So two vertices of B are joined if and only if their labels are disjoint, and the graph is unique.

We see also that the induced subgraph on B is isomorphic to the Petersen graph.

Solution 6 Take the vertices to be the 2-subsets of $\{1, \ldots, 5\}$, and the edges of the first Petersen to be the disjoint pairs of 2-sets. Now trial and error shows that the following is a Petersen graph (I give the adjacency list for each vertex):

12: 14, 15, 23

13: 14, 34, 35

14: 13, 14, 24

15: 12, 35, 45

23: 12, 25, 34

24: 14, 25, 45

25: 23, 24, 35

34: 13, 23, 45

35: 13, 15, 25

45: 15. 24, 34

I leave it to you to calculate the remaining edges and check that they form a bipartite graph.

Solution 7 We saw that the remaining graph is regular with valency 5, and has eigenvalue -3 with multiplicity 5. Let $\lambda_1, \ldots, \lambda_{10}$ be the remaining eigenvalues. Then we have

$$\begin{aligned}\lambda_1 + \cdots + \lambda_{10} &= -(5-15) = 10, \\ \lambda_1^2 + \cdots + \lambda_{10}^2 &= 80 - 25 - 45 = 10,\end{aligned}$$

where the first equality comes from the fact that the adjacency matrix A of the graph has trace 0, and the second because the diagonal entries of A^2 are all 5, so its trace is 80.

Now it follows that

$$(\lambda_1 - 1)^2 + \cdots + (\lambda_{10} - 1)^2 = 0,$$

so $\lambda_1 = \cdots = \lambda_{10} = 1$.

But then our graph is cospectral with the Clebsch graph, and so is isomorphic to it.

Solution 8 We saw that the graphs corresponding to the three cosets are all isomorphic, so it suffices to deal with the case where the connection set is the set F of fifth roots of unity.

Now the non-trivial fifth roots of unity satisy the equation

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

This polynomial is irreducible, so no proper subset of the fifth roots of unity has sum 0. This shows that there can be no cycle of length 3, since such a cycle would have the form $(x, x+\alpha, x+\alpha+\beta, x+\alpha+\beta+\gamma)$, where $\alpha+\beta+\gamma = 0$. Similarly a cycle of length 4 must have the form $x, x+\alpha, x+\alpha+\beta, x+\beta, x)$, and so there are just two paths of length 2 between vertices at distance 2. So the graph is strongly regular with parameters $(16, 5, 0, 2)$.

Solution 9 Suppose that the edges of K_{17} are coloured red, green and blue. Take a vertex v; then v lies on 16 edges, and so by the Pigeonhole Principle, at least six of them have the same colour, without loss of generality red. Now, if two of these six are joined with a red edge, then we have a red triangle; otherwise, we have six vertices with the edges coloured blue and green, so there is a monochromatic triangle in one of these two colours.

Solution 10 The adjacency matrix A satisfies

$$A^2 = kI + (J - I - A),$$

and the all-1 vector is an eigenvector with eigenvalue k. On the space orthogonal to this vector, A satisfies $A^2 + A - (k-1)I = 0$, and so its eigenvalues satisfy $\alpha^2 + \alpha - (k-1) = 0$.

Suppose first that these eigenvalues are irrational. Then they have equal multiplicities, necessarily $k^2/2$ (since there are $k^2 + 1$ vertices altogether); the sum of an eigenvalue and its conjugate is -1. So the trace of A is $0 = k + (k^2/2)(-1)$, whence $k = 2$.

So the eigenvalues are integers, and the discriminant of the quadratic, which is is $4k - 3$, is a perfect square, say $(2s+1)^2$, giving $k = s^2 + s + 1$, and the eigenvalues are s and $-s-1$.

Suppose that their multiplicities are f and g. We have

$$\begin{aligned} f + g &= k^2, \\ sf - (s+1)g &= -k. \end{aligned}$$

Thus $f = s(s^2+s+1)(s^2+2s+2)/(2s+1)$. Since this is an integer, the remainder theorem shows that $2s+1$ divides 15, so $s = 0$, 1, 2 or 7, and $k = 1$, 3, 7 or 57. Of course $k = 1$ is impossible.

Solution 11 First, there are 15 1-factors, as claimed. For the number of ways of choosing three disjoint edges is $\binom{6}{2}\binom{4}{2}\binom{2}{2}$; divide by the number 3! of orders in which they may be chosen.

Now each 1-factor meets six others (two through each of its three edges), so is disjoint from eight others. Two disjoint 1-factors form a 6-cycle, with three long and six short diagonals; it is easy to see that the only ways to make a 1-factor out of these are either take the three long diagonals, or one long and the two short diagonals perpendicular to it. Since we need to use each diagonal once, we must use the three of the second type to complete the 1-factorization. So there are $15 \cdot 8 \cdot 3 \cdot 2 \cdot 1$ choices of five pairwise disjoint 1-factors in order; divide by the number 5! of orders in which they must be chosen.

Solution 12 The 25 cells in different rows and columns from the starting cell make up the 5 neighbours and the 20 at distance 2 of the starting cell, so the distance is at least 3. Now take two cells in the same row, say v and w. Choose a neighbour u of v not in the same column as w; there is a path of length 2 from u to w (they cannot be joined, else u has two neighbours in the same row), so must have distance 2, whence the distance from v to w is 3.

Solution 13 It suffices to show that the diameter is 2. Clearly any two points of the first four types are at distance at most 2, and the same goes for a point of $A \times B$ and one of the other types. We have seen that the only pairs in the Sylvester graph not at distance 1 or 2 are in the same row or column, and so have a common neighbour in either A or B.

Solution 14 By adjusting the alphabets in the squares, we can assume that each has alphabet $\{1, \ldots, n\}$, and that these numbers occur in order in the first row. Where does 1 occur in the second rows of the squares? It cannot occur in column 1, and it cannot occur in the same position in different squares; so there are only $n-1$ places it can occur (at most), and hence at most $n-1$ squares.

Solution 15 Clearly there are n^2 vertices. Any vertex is joined to the $n-1$ vertices in the same row, the $n-1$ in the same column, and the $n-1$ having the same entry in each of the squares; altogether $s(n-1)$.

Take two cells in the same row, say $(1,1)$ and $(1,2)$. There are $n-1$ further cells in row 1. In column 1, we have one cell agreeing with the $(1,2)$ entry in each of the $s-2$ squares; similarly for column 2. Finally, choosing any two squares, there is a unique cell where the $(1,1)$ entry of the first and the $(1,2)$ entry of the second occur. This gives

$$n - 2 + s - 2 + s - 2 + (s-2)(s-3) = s^2 - 3s + n$$

common neighbours. A similar argument holds for two cells having the same entry in one of the squares.

Finally take two cells not in the same row or column and having different entries in all the squares; without loss $(1,1)$ and $(2,2)$. They are joined to cells $(1,2)$ and $(2,1)$; the $s-2$ cells in row 1 having the same entry as the $(2,2)$ cell in one of the squares; the $s-2$ cells in row 2 having the same entry as the $(1,1)$ cell in one of the squares; similarly for columns; and $(s-2)(s-3)$ cells having the same entry as the $(1,1)$ cell in one square and the same entry as the $(2,2)$ cell in another, for each choice of two squares. This gives

$$2 + 4(s-2) + (s-2)(s-3) = s(s-1)$$

common neighbours.

Solution 16 I will use properties of the system without proof, where necessary.

The first thing to notice is that the automorphism group of the graph contains the automorphism group of the Steiner system, which is transitive on the points and on the blocks, so has orbits of length 1, 22 and 77.

Now the strategy is as follows. Choose a point p of the Steiner system. It has 22 neighbours (the point $*$ and 21 blocks containing p) and 77 non-neighbours. Now let P and B be the sets of neighbours and non-neighbours respectively. Identify each element of B with the subset of P consisting of the vertices joined to it. Show that each such subset has size 6, and that any three elements of P are joined to a unique common member of B. Thus we have again a Steiner system $S(3,6,22)$. Moreover, elements of B which are joined have no common neighbours in P (since the graph is triangle-free); counting edges, we see that elements of B with no common neighbours in P must be joined. Thus the graph can be constructed from this new Steiner system in the same way as before, and so it has another group of automorphisms with orbit lengths 1, 22 and 77 (where the orbit of length 1 is the set $\{p\}$). Together these subgroups generate a group of automorphisms which is transitive on all 100 vertices.

Now to verify that the graph is strongly regular, we don't have to check all pairs of vertices, but only those where one of the vertices is $*$; for these the parameters are clear.

Solution 17 I won't do all of this. It is much easier if you use the following fact. Suppose we have a *quasi-symmetric* 2*-design*: that is, a collection of

k-subsets of a v set with the property that any two points lie in a constant number λ of blocks, and any two blocks intersect in either a or b points, for some a and b. Then the graph whose vertices are the blocks, adjacent if they intersect in a points, is strongly regular. To show this, let M be the *incidence matrix* of points and blocks. Then $MM^\top = (r - \lambda)I + \lambda J$, where r is the (constant) number of blocks containing a point. So we know the spectrum of $MM^\top$. By linear algebra, the spectrum of $M^\top M$ is the same apart from the addition or removal of some zeros. But

$$M^\top M = kI + aA = b(J - I - A),$$

where A is the adjacency matrix of the required graph. Using this, one can compute that A has only two eigenvalues on $\mathbf{J}^\perp$, and so that the graph is strongly regular.

Now verify that the assumptions of the preceding paragraph hold in the cases where we take either all the points and blocks of $S(3, 6, 22)$, or we remove a point and use only the blocks not containing the removed point.

Solution 18 I will leave this as a challenge.