Graphs defined on groups

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Brauer and Fowler

In 1955, Richard Brauer and K. A. Fowler published a paper in the *Annals of Mathematics*. With hindsight, this was the first step on the thousand-mile journey to the Classification of Finite Simple Groups.

This paper is remembered for the following result (although they do not state it as a theorem): Given a finite group H with an involution (element of order 2) in its centre, there are only finitely many finite simple groups G containing involutions whose centralisers are isomorphic to H.

This led to the program of classifying simple groups by the centraliser of an involution, a crucial component of CFSG.

The word "graph" doesn't occur in the paper. But, given a finite group G, they defined a metric on the set $G^{\#} = G \setminus \{1\}$ by the rule that $d(x, y) = d$ if the shortest sequence $(x = x_0, x_1, \ldots, x_m = y)$ with $x_{i-1}x_i = x_ix_{i-1}$ has $m = d$.

Of course, this is just the distance in a graph …

The commuting graph

The commuting graph of a group G is the graph with vertex set G, in which vertices x and y are joined if $xy = yx$.

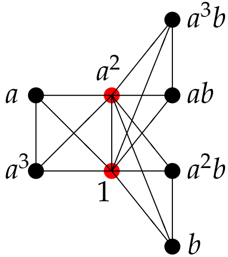
A couple of remarks:

- The definition as stated would give us a loop at every vertex. Sometimes it is important to have these loops; but usually we will silently neglect them.
- Elements in the centre of G, including the identity, are joined to all other vertices. For some questions, such as connectedness, this makes the problem trivial, so (as Brauer and Fowler did) we remove them; for some questions (as we will see in a moment) it is important to keep them; and for yet other questions (such as perfectness) it makes no difference.

I will assume no loops unless I tell you otherwise; if I exclude the centre, I speak of the reduced commuting graph.

An example

Here are the commuting graphs of the two non-abelian groups of order 8: $D_8 = \langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and $Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$.



The two red vertices comprise the centre of the group.

A random walk

Two elements x, y of G are conjugate if $y = g^{-1}xg$ for some $g \in G$. This is an equivalence relation.

Theorem

The limiting distribution of the random walk on the commuting graph (with loops) is uniform on conjugacy classes; that is, the probability of being at x is inversely proportional to the size of the conjugacy class of G containing x.

This is a special case of Mark Jerrum's Burnside process (so called because it is based on the mis-named "Burnside's Lemma"). It is of some practical importance in, for example, computational group theory.

Persi Diaconis has used the idea to argue that the problem of describing conjugacy classes in high-dimensional Heisenberg groups has no nice solution.

The power graph

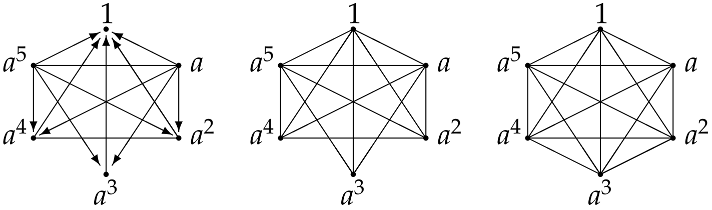
Given a group G, we define the directed power graph of G to have vertex set G, with an arc $x \to y$ if y is a power of x.
From this we define two undirected graphs:

- the power graph, where we ignore directions and multiple edges;
- the enhanced power graph, where we join x and y if there exists z such that $z \to x$ and $z \to y$.

Note that x and y are joined in the enhanced power graph if the group $\langle x, y \rangle$ generated by x and y is cyclic. Compare the commuting graph, where x is joined to y if $\langle x, y \rangle$ is abelian.

An example

The picture shows the direced power graph, power graph, and enhanced power graph of the cyclic group of order 6.



Theorem

For groups G and H, the following are equivalent:

- *the directed power groups are isomorphic;*
- *the power graphs are isomorphic;*
- *the enhanced power graphs are isomorphic.*

However, the directions cannot be uniquely determined, and the three graphs can have different automorphism groups.

The generating graph

A consequence of CFSG is that any non-abelian finite simple group can be generated by two elements. This has sparked a lot of research. These groups G are $1\frac{1}{2}$-generated, meaning that given $x \in G$, there exists $y \in G$ such that $\langle x, y \rangle = G$.

The generating graph of a group G has vertex set G, with an edge from x to y if $\langle x, y \rangle = G$. We say that a graph has spread (at least) k if every k vertices have a common neighbour. Thus $1\frac{1}{2}$-generation is equivalent to having spread 1.

Burness, Guralnick and Harper showed the following. Here the reduced generating graph is the induced subgraph on the non-identity elements.

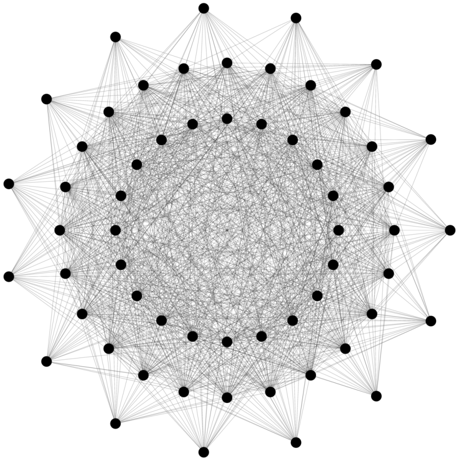
Theorem

For a finite group G, the following are equivalent:

- *the reduced generating graph has spread (at least)* 1*;*
- *the reduced generating graph has spread (at least)* 2*;*
- *any proper quotient of G is cyclic.*

The generating graph of A_5

This beautiful picture was drawn by Scott Harper; I'm grateful to him for permission to use it.



A hierarchy

Rather than isolated results, can we consider these graphs together?
Denoting the power graph, enhanced power graph, commuting graph, and non-generating graph (the complement of the generating graph) of G by $\mathrm{Pow}(G)$, $\mathrm{EPow}(G)$, $\mathrm{Com}(G)$ and $\mathrm{NGen}(G)$, we have the following, where $\subseteq$ means "is a spanning subgraph of":

Proposition

- *For any group G,* $\mathrm{Pow}(G) \subseteq \mathrm{EPow}(G) \subseteq \mathrm{Com}(G)$.
- *If G is non-abelian or not 2-generated, then* $\mathrm{Com}(G) \subseteq \mathrm{NGen}(G)$.

Bojan Kuzma and I devised another graph, the deep commuting graph, which lies between the enhanced power graph and the commuting graph, but I will not discuss it here.

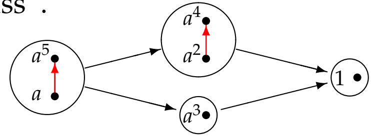
The power graph is perfect

The directed power graph, with a loop at each vertex, is a partial preorder, a reflexive and transitive relation; and the power graph is its comparability graph.

Proposition

The power graph of a finite group is the comparability graph of a partial order.

We simply refine the preorder by totally ordering each "indifference class".



According to Dilworth's Theorem, this implies that the power graph is a perfect graph (that is, all its induced subgraphs have clique number equal to chromatic number).

Universality

Is there something special about the other graphs in the hierarchy?

Theorem

- *If* Γ *is the comparability graph of a partial order, then there is a group* G *such that* Γ *is an induced subgraph of* $\mathrm{Pow}(G)$*.*
- *If* χ *denotes one of* EPow, Com *or* NGen*, then for any finite graph* Γ *there exists a group* G *such that* Γ *is an induced subgraph of* $\chi(G)$*.*

So we need to ask different questions:

- For which finite groups G is, for example, the commuting graph (or one of the others) perfect?
- What is the smallest group such that every n-vertex graph is embedded in its commuting graph (or one of the others)?

Universality, 2

In fact, a stronger result is true:

Theorem

Given an arbitrary colouring of the edges of a finite complete graph red, green and blue, there is a group G and an embedding of the complete graph into G such that

- *the red edges belong to* $\mathrm{EPow}(G)$;
- *the green edges belong to* $\mathrm{Com}(G)$ *but not* $\mathrm{EPow}(G)$;
- *the blue edges do not belong to* $\mathrm{Com}(G)$.

The universality of $\mathrm{EPow}(G)$, $\mathrm{Com}(G)$, and $\mathrm{Com}(G) - \mathrm{EPow}(G)$ are all specialisations of this result. Probably more results of this kind await discovery.

Cographs and twin reduction

A graph Γ is a cograph if it contains no induced 4-vertex path; equivalently, it can be built from 1-vertex graphs by the operations of complementation and disjoint union.
Two vertices in a graph are twins if they have the same neighbours, possibly excluding each other. Twin reduction is the process of identifying twin vertices until no twins remain.

Theorem

- *The result of twin reduction is unique up to isomorphism. (This is the cokernel of the original graph.)*
- *A graph is a cograph if and only if its cokernel is the* 1*-vertex graph.*

All the graphs in our hierarchy have non-trivial twin relation. What can be said about their cokernels? Which are cographs?

The groups $\mathrm{PSL}(2, q)$

Let q be a prime power. If q is a power of 2, let $\{l, m\} = \{q-1, q+1\}$; otherwise let $\{l, m\} = \{(q-1)/2, (q+1)/2\}$.

Proposition

The power graph of $\mathrm{PSL}(2, q)$ is a cograph if and only if each of l and m is either a prime power or the product of two distinct primes.

Are there infinitely many q such that these conditions are satisfied? This is probably a hard number-theoretic problem. The values of d up to 200 for which $\mathrm{Pow}(\mathrm{PSL}(2, 2^d))$ is a cograph are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.

Automorphism groups

For each type X of graph in the hierarchy, the automorphism group of G acts by automorphisms of $X(G)$. You might expect that there is not too much more.

However, if you compute the number of automorphisms of the power graph of A_5, you come up with the absurdly large number

$$668594111536199848062615552000000.$$

We can make two observations:

Proposition

- *For any type* X *in the hierarchy, if* $G \neq \{1\}$, *then* $\mathrm{Aut}(X(G))$ *has a non-trivial normal subgroup which is the direct product of symmetric groups on the twin classes.*
- *If* Γ *is a cograph, then* $\mathrm{Aut}(\Gamma)$ *is built from the trivial group by the operations of direct product and wreath product with symmetric groups.*

Automorphism groups, 2

However, sometimes interesting things turn up.
If we start with the power graph of the Mathieu group M_{11}, remove the identity, and apply twin reduction, we get a graph on 1210 vertices with automorphism group M_{11}, acting with just four orbits.
Digging into this graph, we find an interesting bipartite graph, also with automorphism group M_{11}, which is bipartite with bipartite blocks of sizes 165 and 220, and semiregular with valencies 4 and 3; it has diameter and girth equal to 10.
So there are jewels among the dross, if you look in the right place!

The Gruenberg–Kegel graph

The Gruenberg–Kegel graph (GK graph for short), sometimes called the prime graph, of a finite group G is the graph whose vertex set is the set of prime divisors of G, with vertices p and q joined by an edge if G contains an element of order pq. It can be tiny compared to $|G|$.

This was introduced by Gruenberg and Kegel to study indecomposability of the augmentation ideal of the integral group ring of G. They proved but did not publish a structure theorem for groups whose GK graph is disconnected. The theorem was published by Williams (a student of Gruenberg).

GK graphs are now the subject of intensive research.

Natalia Maslova and I proved that there is a function F such that if a graph on n vertices (labelled by primes) is the GK graph of more than $F(n)$ groups, then it is the GK graph of infinitely many. Our function was $O(n^7)$; this is probably not best possible.

The GK graph and the hierarchy

Theorem
Let X be one of Pow, EPow *or* Com. *If G and H are groups with $X(G) \cong X(H)$, then G and H have the same GK graph.*

Theorem
Let G be a group with $Z(G) = \{1\}$. Then the reduced commuting graph of G is connected if and only if the GK graph of G is connected.

Theorem
Let G be a finite group. Then $\mathrm{Pow}(G) = \mathrm{EPow}(G)$ *if and only if the GK graph of G is a null graph.*

Diameter of the reduced commuting graph

After a lot of research, Iranmanesh and Jafarzadeh conjectured that there is an absolute upper bound on the diameter of a connected component of the reduced commuting graph.
This was refuted by Giudici and Parker; however Morgan and Parker proved it for graphs with trivial centre:

Theorem

- *For every d, there is a group whose order is a power of 2, whose reduced commuting graph has diameter greater than d.*
- *If G is a group with $Z(G) = \{1\}$, then every component of the reduced commuting graph of G has diameter at most 10.*

Want to know more?

There is a huge amount that I haven't said. If you are interested to know more,

- I have a long paper in the *International Journal of Group Theory*; download it (free) from `https://ijgt.ui.ac.ir/article_25608.html` .
- On 8 and 9 June, I am giving a (virtual) intensive course at the London Taught Course Centre (8 hours of lectures over a 24-hour period). Register (free) by email to `office@ltcc.ac.uk` .

. . . for your attention.