Generalisations of EPPO groups using graphs

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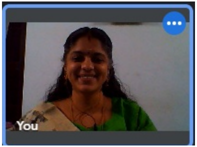

The organisers were Vijayakumar Ambat and Aparna Lakshmanan at CUSAT, to whom I am grateful.

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- Of the two groups of order 6, the dihedral group is an EPPO group (all elements have orders 1, 2 or 3) but the cyclic group is not.
- Thinking about this example, we see that a nilpotent group (which is the direct product of its Sylow subgroups) is an EPPO group if and only if it has prime power order.

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Earlier this year, I asked Natalia Maslova if she knew a classification of all EPPO groups. She sat down and produced one. I will tell you later why I wanted this.

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- $|\pi(G)| = 3$, $G/O_2(G)$ *is* $\mathrm{PSL}_2(2^n)$ *for* $n \in \{2, 3\}$ *and if* $O_2(G) \neq \{1\}$, *then* $O_2(G)$ *is the direct product of minimal normal subgroups of G, each of which is of order* 2^{2n} *and as* $G/O_2(G)$*-module is isomorphic to the natural* $\mathrm{GF}(2^n)\,\mathrm{SL}_2(2^n)$*-module.*

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- $|\pi(G)| = 4$ *and* $G \cong \mathrm{PSL}_3(4)$.
- $|\pi(G)| = 4$, $G/O_2(G)$ *is* $\mathrm{Sz}(2^n)$ *for* $n \in \{3, 5\}$*, and if* $O_2(G) \neq \{1\}$*, then* $O_2(G)$ *is the direct product of minimal normal subgroups of G, each of which is of order* 2^{4n} *and as* $G/O_2(G)$*-module is isomorphic to the natural* $\mathrm{GF}(2^n)\,\mathrm{Sz}(2^n)$*-module of dimension* 4*.*

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- A glance at the $\mathbb{ATLAS}$ of finite groups shows, for example, that the Mathieu group M_{11} has vertex set $\{2, 3, 5, 11\}$ and just a single edge $\{2, 3\}$.
- G is an EPPO group if and only if its GK graph is a null graph (that is, has no edges).

Frobenius and 2-Frobenius groups

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The symmetric group S_4 is an example.

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Let G be a finite group whose GK graph is disconnected. Then one of the following holds:

- *G is a Frobenius or 2-Frobenius group;*
- *G is an extension of a nilpotent π-group by a simple group by a π-group, where π is the set of primes in the connected component containing 2.*

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Which simple groups can occur in the second conclusion of the theorem? This was investigated by Williams, though he was unable to deal with groups of Lie type in characteristic 2. The work was completed by Kondrat'ev in 1989, and some errors corrected by Kondrat'ev and Mazurov in 2000.

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I turn now to some other ways to generalise the EPPO groups using graphs.

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We see that the edge set of $\mathrm{Pow}(G)$ is contained in that of $\mathrm{EPow}(G)$. (In graph theory language, $\mathrm{Pow}(G)$ is a spanning subgraph of $\mathrm{EPow}(G)$.)

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Conversely, if G is an EPPO group, and $\langle g, h\rangle$ is cyclic, then it has prime power order, and so one of g and h generates this group, say g; then h is a power of g. □

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Thus the classification of EPPO groups gives us the groups G for which $\mathrm{Pow}(G) = \mathrm{EPow}(G)$.

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Clearly this class of groups includes the EPPO groups!

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(The Catalan conjecture asserts that the only solution of $x^a - y^b = 1$ in positive integers x, y, a, b with $a, b > 1$ is $3^2 - 2^3 = 1$. It was proved by Mihăilescu in 2002.)

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The strategy of the proof is to show that, given a matching in the enhanced power graph, we can replace its edges by edges of the power graph to find another matching of the same size.

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Since the path P_4 is isomorphic to its complement, the class of cographs is self-complementary. In fact, it is the smallest class of graphs containing the 1-vertex graph and closed under disjoint union and complementation. This means that the class has very nice algorithmic properties, which don't concern us here.

The power graph of a p-group is a cograph

Recall that in the power graph, g and h are joined if one is a power of the other. So the graph is naturally a directed graph, with an arc $g \to h$ if h is a power of g. It is easily seen that this relation is transitive.

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Now in a p-group, if $c \to b$ and $c \to d$, then b and d lie in a cyclic group of prime power order, so one is a power of the other. Hence there can be no induced P_4:

Theorem

The power graph of a group of prime power order is a cograph.

The power graph of an EPPO group is a cograph

This follows easily from the previous result. Hence the following problem is a generalisation of the problem of determining EPPO groups:

Problem

Determine the finite groups whose power graph is a cograph.

I have worked on this problem with Pallabi Manna and Ranjit Mehatari from Rourkela. Our first theorem states:

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If G is a nilpotent group, then the power graph of G is a cograph if and only if either G has prime power order, or $G = C_{pq}$ where p and q are primes.

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Recall that a nilpotent EPPO group has prime power order. The addition of the groups C_{pq} has a big effect on the class of groups!

Simple groups whose power graph is a cograph

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- $G = \mathrm{Sz}(q)$ *for q an odd power of 2, where each of $q-1$,* $q+\sqrt{2q}+1$ *and* $q-\sqrt{2q}+1$ *is either a prime power or the product of two primes;*

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Note that $\mathrm{PSL}(2,11)$ and M_{11} have identical GK graphs, but the power graph of the first is a cograph, that of the second is not.

A problem for number theorists

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For example, the values of d up to 200 for which the power graph of $\mathrm{PSL}(2,2^d)$ is a cograph are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.

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Similar (possibly easier) question for $\mathrm{Sz}(q)$.

More graphs

I will finish with a couple of recent results, not specifically about EPPO groups, but about the question: given two graphs on a group, for which groups do they coincide? We will see that other interesting classes of groups arise.

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We saw that the enhanced power graph is equal to the power graph if and only if G contains no $C_p \times C_q$ where p and q are distinct primes. Similarly, the commuting graph equals the enhanced power graph if and only if G contains no $C_p \times C_p$, where p is prime.

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From this, the groups can be determined.

Let G be a group containing no $C_p \times C_p$. By a theorem of Burnside, all Sylow subgroups of G are cyclic or generalized quaternion.

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- a cohomological argument due to Glauberman shows that any such group H/Z has a unique cover H with unique subgroup of order 2;
- using the fact that the other Sylow subgroups are cyclic, it is possible to determine G.

Super graphs

Following work by several authors, G. Arunkumar, Rajat Kanti Nath, Lavanya Selvaganesh and I defined, for each type of graph on a group G, a conjugacy supergraph, in which g and h are joined if and only if there are conjugates of g and h which are joined in the original graph.

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Here are two theorems from our paper in preparation.

Two theorems

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Theorem

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- *G is a Dedekind group, that is, every subgroup is normal.*

The independence graph

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The generating graph of a group G has an edge from g to h whenever $\langle g, h \rangle = G$. It has been the subject of much research. Unfortunately, if G is not 2-generated, then the generating graph is null. To overcome this defect, Lucchini defined the independence graph of G, in which g and h are joined if and only if $\{g, h\}$ is contained in a minimal (with respect to inclusion) generating set for G.

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Note that, if h is a power of g, then $\{g, h\}$ is not contained in any minimal generating set. In a paper in preparation, Lucchini and Nemmi say that G has the independence property if the converse holds, that is, the independence graph is the complement of the power graph.

Lucchini and Nemmi determined the soluble groups with the independence property. They also showed that there are no non-soluble groups, using the following very recent theorem of Saul Freedman:

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Theorem

Let S be a non-abelian finite simple group. Then there exist non-commuting elements $s, x \in S$ such that, whenever G is an almost simple group with socle S, and $\mathcal{M}_G(s)$ denotes the set of maximal subgroups of G containing s, then

$$x \in \bigcap_{M \in \mathcal{M}_G(s)} M.$$

Another variation is to define the rank graph to have an edge $\{g, h\}$ whenever $\{g, h\}$ is contained in a generating set of minimal cardinality for G, this minimal cardinality being the rank of G. If the rank is 2, this is just the generating graph.

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Just as edges of the power graph cannot be joined in the independence graph, so edges of the enhanced power graph cannot be joined in the rank graph. So, as above, we call G rank perfect if the rank graph is the complement of the enhanced power graph.
Lucchini has shown that a rank perfect group must be supersoluble, and classified the non-nilpotent groups with this property.

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… for your attention.