Finding geometries in the power graphs of simple groups

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I hope you enjoy hearing about the search.

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The original and most famous of these is the commuting graph of a group G, whose vertices are the non-central elements of the group, two vertices x and y joined if $xy = yx$. This was introduced by Brauer and Fowler in 1955, and their results on its diameter (for a non-abelian simple group G) led to a bound on $|G|$ in terms of the order of an involution centraliser, arguably the first step towards the Classification of Finite Simple Groups.

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Another famous graph is the generating graph, in which x and y are joined if $\langle x, y \rangle = G$. It was introduced by Liebeck and Shalev and has been important in studying probabilistic statements about generating finite simple groups. (Any such group can be generated by two elements.)

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Then Ambat Vijayakumar from Kerala, in south India, invited me to help run an on-line discussion on "Graphs and groups". This ran for six months, and resulted in a huge amount of new mathematics, which is still being produced. What I am talking about here is a very small part of the whole.

The power graph

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I will use it here because it sits at the bottom of a hierarchy of graphs, and so tends to be relatively sparse for interesting groups.

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If v and w are twins, then the permutation interchanging them and fixing the other vertices is an automorphism. So graphs with many twins have huge automorphism groups, mostly uninteresting rubbish.

Twin reduction and the cokernel

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If necessary I will silently delete isolated or dominating vertices (there can be at most one of these in the cokernel).

Cographs

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A graph Γ *is a cograph if and only if its cokernel is the* 1*-vertex graph.*

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In the first three cases, deciding which values of q occur seems to be a problem beyond the current reach of number theory!

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I do not know why the components in the second and third case are the same.

More interesting cases

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In each of these three cases, the automorphism group of the graph is equal to the automorphism group of the group.

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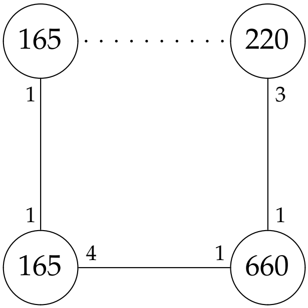

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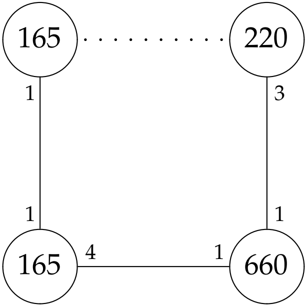

From this we can build a bipartite graph on $165 + 220$ vertices, where the vertices in the two parts have valencies 4 and 3 respectively.

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Since it is bipartite, it is presumably the incidence graph of a nice geometry with 165 points and 220 lines, having automorphism group M_{11}. Two points lie on at most one line, and there are no triangles or quadrilaterals. I am not sure whether this geometry is already known, or what other properties it may have.

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I suspect that similar beautiful objects can be extracted from other finite simple groups in a similar way.

Some speculations

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For which finite simple groups is the cokernel of the power graph (less isolated vertex) connected? In particular, is this the case for most groups of Lie type with rank greater than 1*, and for most sporadic groups?*

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What happens for other graphs defined on groups?

… for your attention.