Resistance distance and resistance distance transform

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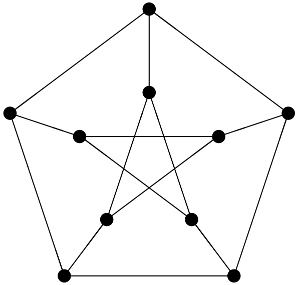
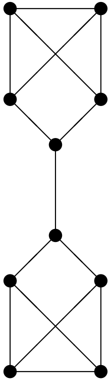
My main interest here is in the second question, but you will see that the two questions are more closely related than you might first think.

What makes a good network?





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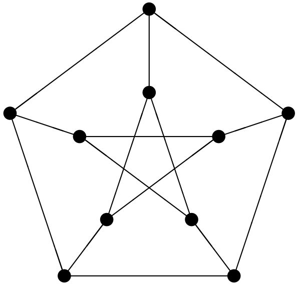

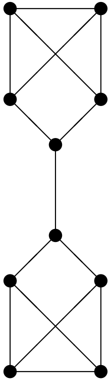

Suppose you wanted to build a network connecting ten nodes. You could afford to construct fifteen edges. You want the network to be well connected and resilient. Which one of the two shown above would you choose?

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But what if we are faced with huge networks which cannot be drawn in a simple way?

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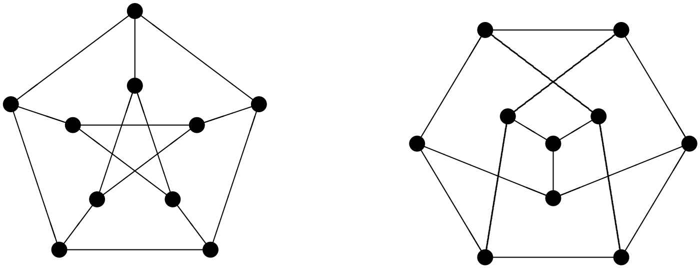

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This is the famous graph isomorphism problem, one of a select class of problems in the complexity class NP which are not known to be either in P (polynomial-time solvable) or NP-complete (equivalent to the hardest problems in NP).
In the last decade, László Babai found an algorithm for graph isomorphism which runs in quasipolynomial time, that is, time bounded by $O(\exp(a(\log n)^c)$ for some constants a and c. (This is polynomial if $c = 1$.)

Coherent configurations

One of the pioneers on the graph isomorphism problem was Boris Weisfeiler, who worked on it in the former Soviet Union in the 1960s. He emigrated to the USA in the 1970s, then in the 1980s went hiking in Chile; he disappeared, and no trace of him has ever been found.

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Weisfeiler and his colleague Leman devised an algorithm which, given a graph, constructs a canonical refinement of it, an object which they called a cellular algebra. If two graphs are isomorphic, then these cellular algebras are isomorphic; since they usually have much more structure, it is simpler in practice to test this.

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Weisfeiler's ideas are deeply embedded in Babai's proof.

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Now the term "cellular algebra" has been used in a different context, so these objects are now called "coherent configurations".

The definition

A coherent configuration is a collection of binary relations $R_1, \ldots, R_r$ on a set Ω satifying certain properties, which follow. Associating a colour with each relation, we can think of this as an edge-coloouring of the complete directed graph with loops on Ω.

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- Given $i, j, k \in \{1, \ldots, r\}$ and $(x, y) \in R_i$, the number of $z \in \Omega$ such that $(x, z) \in R_j$ and $(z, y) \in R_k$ depends only on (i, j, k) and not on the choice of x and y.

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The number r is called the rank of the configuration. If all the relations R_i are symmetric, the configuration is an association scheme.

The Weisfeiler–Leman algorithm

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Given $\Pi = \{R_1, \ldots, R_s\}$, regard it as a collection of edge-coloured digraphs on Ω (one for each relation). Now, for each triple (i, j, k), and each choice of $(x, y) \in R_i$, count the number of z for which $(x, z) \in R_j$ and $(z, y) \in R_k$. In general these numbers will not be constant. So refine the partition by splitting R_i into a number of parts, so that these numbers are constant on each part.

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Then iterate this construction. It stabilises after finitely many steps, and the stable partition is clearly a coherent configuration.

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This idea is also related to the question of good networks, as we will see.

Resistance

Given a graph on the vertex set Ω, we can regard it as an electrical network. The simplest way to do this is to put a 1-ohm resistor on each edge of the graph. Then we can measure the effective resistance between any two vertices by connecting those vertices to a 1-volt battery and measuring the current I which flows: the effective resistance is then $1/I$.

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- Ohm's law: potential difference is equal to current times resistance.

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Thus in our two example networks, the average resistance in the Petersen graph is $11/15$; that in the other network is $206/135$, more than twice as high.

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- With weights as before, the effective resistance between v and w is equal to the sum of the weights of 2-component spanning forests with v and w in different components, divided by the sum of the weights of the spanning trees.

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An advantage of this method is that it "sees" the whole graph at each step, unlike the WL algorithm which only looks locally. In many cases they found that the partition was stable after just one step.

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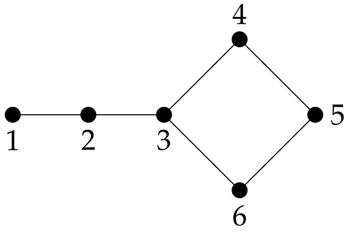

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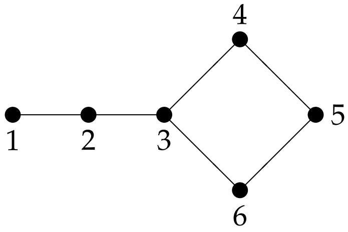
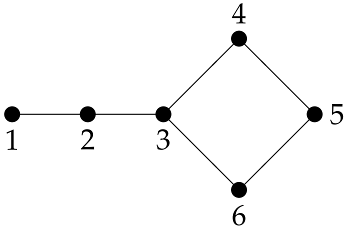
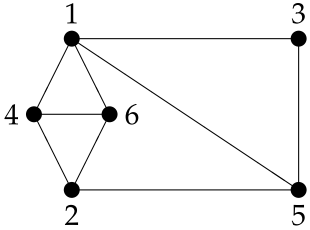


$\{2, 3\}$ is an edge; $\{4, 6\}$ is a non-edge; but $R_{23} = R_{46} = 1$.
In this case, the partition defined by resistance distance does not refine the original partition

Drawbacks, 2





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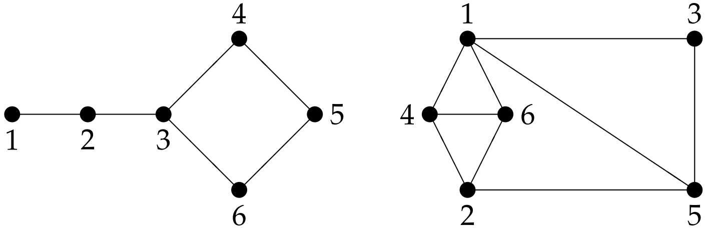

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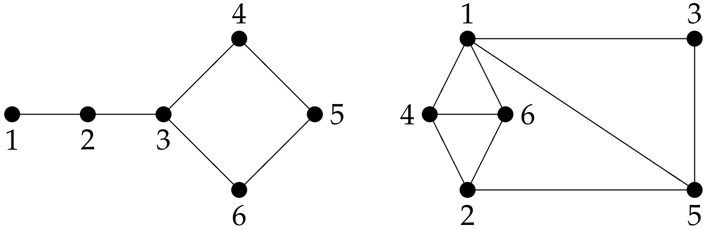

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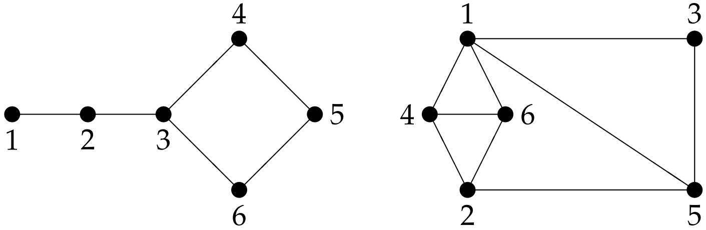

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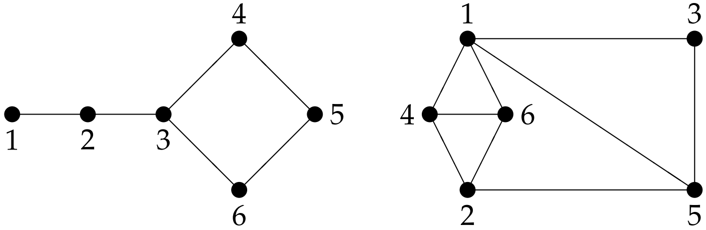

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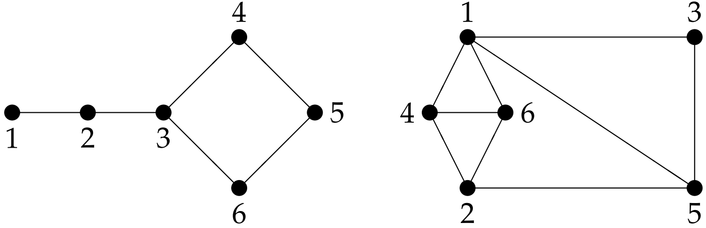

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But we would like a graph and its complement to reduce to the same configuration since they are the same partition of Ω^2.

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Conjecture

Using RDT2, the output partition of each stage refines the input partition; hence the procedure is guaranteed to terminate.

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Thus (and for us this is the important example) the set of real symmetric matrices, with the operation $A * B = \frac{1}{2}(AB + BA)$, is a Jordan algebra.

Jordan schemes

We define a Jordan scheme to be an object satisfying similar axioms to those of a coherent configuration, a collection of binary relations $R_1, \ldots, R_r$ on a set Ω satifying:

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- All the relations are symmetric.
- Given $i, j, k \in \{1, \ldots, r\}$ and $(x, y) \in R_i$, the number of $z \in \Omega$ such that $(x, z) \in R_j$ and $(z, y) \in R_k$ plus the number such that $(y, z) \in R_j$ and $(z, x) \in R_k$ depends only on (i, j, k) and not on the choice of x and y.

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The span over the real numbers of the matrices of a Jordan scheme is thus a Jordan algebra. One can define a Jordan WL algorithm which will produce the largest Jordan scheme below a given partition.

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Mikhail Muzychuk, Sven Reichard, and Mikhail Klin have produced infinite families of examples of proper Jordan schemes.

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And we hope that RDT2 will stabilise in significantly fewer steps than the Jordan WL algorithm! We also hope that new examples of Jordan schemes can be produced in this way. We have made some progress towards these conjectures but they are not yet established.

… for your attention.