Classes of groups defined by graphs

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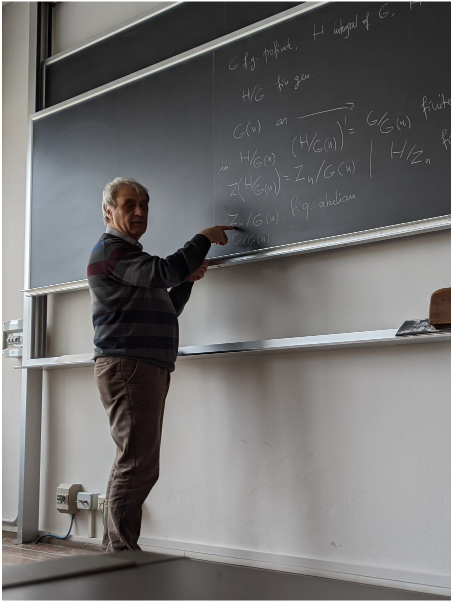
Group Theory in Florence III
1 July 2022

Carlo Casolo, 1958 – 2020



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In February 2020, it was arranged that Carlo, Francesco and I would meet in Florence to work on the sequel. Circumstances were against us: the north of Italy was already in Covid lockdown, though Florence was still open; a storm on 9 February caused my flight to be cancelled, so I arrived a day late; and a storm on 15 February nearly did the same for my return flight (it was one of the bumpiest landings I have experienced). Despite this, it was a greatly successful visit; we proved substantial results for the second paper.

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Before we could get the paper written, Carlo was no longer with us. We felt his absence severely and were only partly successful in understanding his notes. Nevertheless, with Claudio Quadrelli’s help, the paper was written, and has been accepted for publication.

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You can probably think up many group-theoretic constructions which give rise to inverse problems. Some are trivial (for example, G is the centre of some group if and only if G is abelian); some don't make much sense. But the integrability problem turns out to lie in a sweet spot where the theory is non-trivial and interesting.

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Nevertheless, they have a lot to say to one another.

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The first such graph is the commuting graph, introduced by Brauer and Fowler in their seminal 1955 paper: two elements of G are joined by an edge if they commute. This paper showed that there are only finitely many finite simple groups of even order with a prescribed involution centraliser; this could be said to be the first step towards the classification of the finite simple groups.

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In fact Brauer and Fowler don't use the word "graph" in the paper; but their main tool is the graph distance in the induced subgraph on the non-identity elements, and the main use they make of it is to show that the diameter of this graph is surprisingly small.

Outline of the talk

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Since I am more interested in comparing graphs than in their specific properties, I will assume in all cases that the vertex set is the group G. People who investigate properties of the graphs often vary this: for example, in studies of the commuting graph, it is common to remove the centre of G (the set of dominating vertices in the graph).

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Clearly the recipe in the last two cases can be extended; we can define the nilpotency graph (resp. solubility graph) by the joining rule that $\langle g, h \rangle$ is nilpotent (resp. soluble).

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These properties are straightforward to prove, and lead to complete classifications of the relevant classes of groups, as I will indicate.

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The relevance for us is that a group G has power graph and enhanced power graph equal if and only if its GK graph has no edges; so, if G is not a p-group, then the Gruenberg–Kegel theorem applies. Such groups, in which all elements have prime power order, are known as EPPO groups.

EPPO groups

EPPO groups were first investigated by Higman in the 1950s, in the soluble case. In the 1960s, Suzuki, in the course of discovering his infinite family of simple groups, determined all the simple EPPO groups (only certain $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$ and the group $\mathrm{PSL}(3, 4)$). A complete determination was given by Brandl in 1981, in a paper which is somewhat inaccessible; not surprisingly, this result has been rediscovered several times.

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A modern account is in my paper with Natalia Maslova in the *Journal of Algebra*, which includes also a survey of GK graphs and the extent to which a group is determined by its GK graph.

Enhanced power graph and commuting graph

These two graphs are equal if and only if there are no subgroups $C_p \times C_p$. A theorem of Burnside shows that the Sylow subgroups must be cyclic or (for $p = 2$) generalised quaternion. Now theorems of Burnside, Gorenstein and Walter, and Glauberman give a complete classification.

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Bojan Mohar and I introduced a graph lying between the enhanced power graph and the commuting graph, which we called the deep commuting graph. Two elements are joined if and only if their inverses in every central extension of G commute. Now we have the further problems of deciding when this graph can coincide with either of its neighbours. This involves more complicated machinery such as the Bogomolov multiplier of G. I will not discuss it here.

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Our paper is to appear in the *Journal of Graph Theory*.

Super graphs

The second class of graphs consist of super graphs, defined as follows: we have a graph on G (typically invariant under $\mathrm{Aut}(G)$), and an equivalence relation on G (also $\mathrm{Aut}(G)$-invariant); we join g and h if there exist g' and h', in the same equivalence classes as g and h, which are joined in the graph. It would be natural to take the vertex set to be the set of equivalence classes; but since I am comparing graphs, I will stick with the whole group as vertex set.

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This gives us several more cases where we can ask about equality; I will deal with just two. The others are open problems.

Graph and supergraph

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The class of 2-Engel groups includes the 2-nilpotent groups, and is included in the 3-nilpotent groups (a theorem of Hopkins and Levi, independently).

The generating graph and its relatives

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All non-abelian finite simple groups are generated by two elements, and the generating graph is an important tool for studying these. However, if a group G cannot be generated by two elements, then the generating graph has no edges.

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When does equality hold?

Independence and perfect rank properties

This question has recently been answered by Saul Freedman, Andrea Lucchini, Daniele Nemmi, and Colva Roney-Dougal. They say that groups in which the independence graph is the complement of the power graph have the independence property, while groups in which the rank graph is the complement of the enhanced power graph have the perfect rank property.

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Theorem

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The precise description is fairly long, and I will leave it out; the paper should be available on the arXiv soon. Perfect rank takes only a couple of pages, but independence is much harder and involves detailed knowledge of the finite simple groups.

Invariable generation

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Question

For which groups does equality hold in each of the above three inclusions?

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Proposition

Let G be a non-abelian group, and $g \notin Z(G)$. Then there exists $h \in G$ such that g and h invariably don't commute.

The Jordan trick

I have not thought about these questions – they may be easy or impossible! But here is a simple trick that may be useful:

Proposition

Let G be a non-abelian group, and $g \notin Z(G)$. Then there exists $h \in G$ such that g and h invariably don't commute.

It is enough to find h such that no conjugate of h commutes with g. By Jordan's theorem, there exists a conjugacy class disjoint from $C_G(g)$.

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… for your attention.