Graphs defined on groups: the big picture

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Groups and graphs

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Nevertheless, they have a lot to say to one another.

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The first such graph to be considered is the commuting graph: the vertex set is G, and g and h are joined if $gh = hg$.

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In more detail, the centraliser of an element $g \in G$ is the set of elements of G that commute with g; it is a subgroup of G, denoted by $C_G(g)$, and is in fact the closed neighbourhood of g in the commuting graph. An involution is an element of order 2.

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Brauer and Fowler didn't know that a non-abelian finite simple group necessarily contains an involution; the proof of this conjecture of Burnside (by Feit and Thompson) was still nearly a decade in the future when they wrote.

Involution centralisers and CFSG

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As a footnote, Brauer and Fowler don't use the word "graph" in the paper; but their main tool is the graph distance in the induced subgraph on the non-identity elements, and the main use they make of it is to show that the diameter of this graph is surprisingly small, which leads to their bound.

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Of course, if G cannot be generated by two elements, then this is not a very interesting graph. Also, if G is not cyclic, then the identity cannot occur in any 2-element generating set, so is an isolated vertex. For this reason, many authors remove it. To keep things uniform (the vertex set should be G, for comparison with other graphs), I will not do so.

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The generating graph was introduced by Guralnick and Kantor, and has been very useful in questions about probabilistic generation of finite simple groups. (As a consequence of CFSG, we know that all non-abelian finite simple groups are 2-generated.

Generating graph and spread

The spread of a graph is the largest number s such that any s vertices have a common neighbour. Thus, "spread ≥ 1" means no isolated vertices, while "spread ≥ 2" is stronger than "diameter 2", so is much stronger than spread 1. However, the following remarkable result was proved by Burness, Guralnick and Harper recently:

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Theorem

Let Γ be the generating graph of a group G, with the identity removed. Then the following are equivalent:

- Γ *has spread* ≥ 1;
- Γ *has spread* ≥ 2;
- *every proper quotient of G is cyclic.*

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Moreover, the generating graph is equal to the complement of the commuting graph if and only if G is a minimal non-abelian group, that is, a non-abelian group of which every proper subgroup is abelian.

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Moreover, the generating graph is equal to the complement of the commuting graph if and only if G is a minimal non-abelian group, that is, a non-abelian group of which every proper subgroup is abelian.
Minimal non-abelian groups were classified by Miller and Moreno in 1904.

What is in between?

If G is a group which is not abelian and non minimaml non-abelian, then we can consider the non-commuting, non-generating graph of G, in which (as the rather unwieldy name suggests, two elements are joined if they do not commute but do not generate the group).

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This graph was investigated by Saul Freedman in his recent PhD thesis at the University of St Andrews. He was able to prove strong results about its connectedness and diameter (when restricted to non-isolated vertices).

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  - Pick a graph type (e.g. the commuting graph), and ask: for which groups does this belong to a particular graph class (e.g. perfect graphs, cographs)?
  - Pick two graph types, and ask for which groups they are equal or complementary. We saw an example already, the minimal non-abelian groups.

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- The enhanced power graph of G, a variant where g and h are joined if there exists an element k such that g and h are both powers of k.

It is clear that the power graph is contained in the enhanced power graph (as a spanning subgraph).

Also, g and h are joined in the enhanced power graph if and only if $\langle g, h \rangle$ is a cyclic group. Noting that g and h are joined in the commuting graph if and only if $\langle g, h \rangle$ is abelian, we see that the enhanced power graph is a spanning subgraph of the commuting graph.

When does equality hold?

Theorem

- *The power graph of G is equal to the enhanced power graph if and only if G contains no subgroup isomorphic to* $C_p \times C_q$*, where p and q are distinct primes.*

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The proofs are not difficult. If g and h have orders p and q and commute, they are joined in the enhanced power graph but not in the power graph; if they both have order p and commute, they are joined in the commuting graph but not the enhanced power graph. The converse statements are similar.

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More on this shortly.

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They also gave a structure theorem for groups with disconnected GK graph in an unpublished manuscript. The theorem was subsequently published by Gruenberg's student Williams, and later refined by other (mainly Russian) mathematicians.

EPPO groups

The finite group G is an EPPO group if every non-identity element of G has prime power order.

This class was introduced by Graham Higman, who classified the soluble ones, in the 1950s. Michio Suzuki in the 1960s determined the simple EPPO groups (as a spin-off from his construction of an infinite family of finite simple groups). A description of all EPPO groups was published by Rolf Brandl in 1981. It was in a rather obscure journal, with the result that the work was re-done later by several different authors. Now we have a good understanding of these groups.

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Theorem

For a finite group G, the following conditions on G are equivalent:

- *G is an EPPO group;*
- *the GK graph of G has no edges;*
- *the power graph and enhanced power graph of G are equal.*

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If the Sylow 2-subgroups are generalized quaternion, then a specic quotient of G has dihedral Sylow 2-subgroups, and so is described by the Gorenstein–Walter theorem. The other cyclic subgroups restrict the group further, so again the possible groups can be described.

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In the other case, something different happened. Bojan Kuzma and I have defined a graph we call the deep commuting graph, which lies between the enhanced power graph and the commuting graph. Two elements g and h are joined in the deep commuting graph if and only if their inverse images in any central extension of G (that is, any group H with an epimorphism to G whose kernel is contained in the centre of H) commute. Its study involves new ideas: Schur multiplier, isoclinism, Bogomolov multiplier … Our paper will appear shortly in the *Journal of Graph Theory*.

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Now it is relatively easy to show that if two groups of the same order are isoclinic, then their commuting graphs are isomorphic. Is the converse true? Vikramin Arvind and I conjecture that this is so for nilpotent groups of class 2. It fails for groups of class 3. One of the groups in this example is also the smallest group whose deep commuting graph lies strictly between the enhanced power graph and the commuting graph.

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In 1903, Landau proved:

Theorem
Given a positive integer k, there are only finitely many finite groups with just k conjugacy classes.

Since every talk should contain a proof, I will show you the proof of this.

Proof

Proof.
Let $x_1, \dots, x_k$ be conjugacy class representatives. Then by the Orbit–Stabiliser Theorem, $|x_i^G| = |G|/|C_G(x_i)|$. These class sizes sum to $|G|$; so, if $n_i = |C_G(x_i)|$, we have

$$\sum_{i=1}^{k} \frac{1}{n_i} = 1.$$

This equation has only finitely many solutions [Exercise!], and in any solution $(n_1, \dots, n_k)$, the largest n_i is $|C_G(1)| = |G|$. □

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I will show you a different kind of extension.

The SCC graph of a finite group

The soluble conjugacy class graph (for short, the SCC-graph) of G is the graph whose vertex set is the set of conjugacy classes in G, with an edge from x^G to y^G if and only if there exist $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is a soluble group.

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There are numerous variants of the definition: we could replace "soluble" by "nilpotent", "abelian", "cyclic", etc.; and there are other variants possible too.

A theorem

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The proof requires the Classification of Finite Simple Groups, but only in a rather "light-touch" way.

Some history

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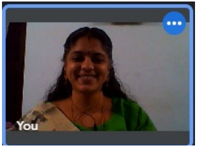

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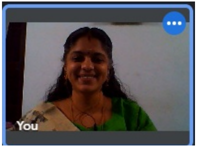

Please join me in exploring further this fascinating topic!

… for your attention.