Graphs defined on groups, 2: Some details

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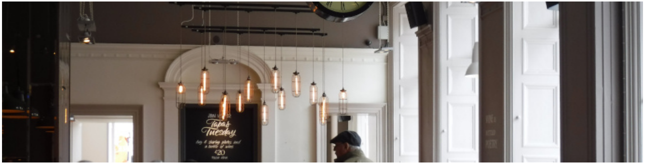

GEARS, Edinburgh, 9 August 2022

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- the nilpotency graph: $\langle g, h \rangle$ nilpotent;
- the solubility graph: $\langle g, h \rangle$ soluble.

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- further results on when two of the graphs are equal or complementary;
- twins, twin reduction, cographs, and finding interesting graphs inside the power graph;
- results on clique number and chromatic number, and a new constant.

Conjugacy class graphs

Here is a variant we have already met. For each of the above types, there is a corresponding conjugacy class graph, whose vertices are the conjugacy classes in G, two vertices C, D joined if there exist $g \in C$ and $h \in D$ such that g and h are joined in the original graph.

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For brevity I will call the expanded X conjugacy class graph the "super X graph" of the group G. The super X graph contains the X graph as an induced subgraph.

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I didn't know before doing this work that a group G is 2-Engel if and only if every centraliser is normal in G.

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- The independence graph of G is the graph with vertex set G, in which g and h are joined if and only if $\{g, h\}$ is contained in a minimal (with respect to inclusion) generating set of G.
- The rank graph of G is the graph with vertex set G, in which g and h are joined if and only if $\{g, h\}$ is contained in a generating set of minimum cardinality.

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When does equality hold?

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However, the proof requires CFSG together with a very detailed knowledge of the finite simple groups, including results about the set of all maximal subgroups containing a given element, and the authors had to correct some statements in the literature in the course of this.

Invariable generation

A set $\{g_1, \ldots, g_n\}$ of elements of G invariably generates G if we can replace each g_i by an arbitrary conjugate and still have a generating set; that is, $\{g_1^{x_1}, \ldots, g_n^{x_n}\}$ is a generating set for arbitrary $x_1, \ldots, x_n \in G$.

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It is also possible to define invariable analogues of the independence and rank graphs, which bear similar relationships to the superpower graph and superenhanced power graph. I think these questions haven't even been considered!

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What is going on?

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Random graphs don't have twins, but graphs from groups typically do. For example, in the power graph, two elements which generate the same cyclic subgroup are twins. So, in A_5, we have a subgroup $S_4^6 \times S_2^{10}$ of such automorphisms, which are really of no interest.

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But maybe we shrink it to a single vertex ...

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Problem

For which groups is the power graph a cograph?

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In the first three cases, deciding which values of q occur seems to be a problem beyond the current reach of number theory!

Some number theory

There is some hard number theory lurking in the above, namely the problem of deciding when q satisfies the conditions of the theorem:

- For which q (a power of 2) are $q + 1$ and $q - 1$ each either a prime power or the product of two primes?
- For which q (an odd prime power) are $(q + 1)/2$ and $(q - 1)/2$ each either a prime power or the product of two primes?
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- For which q (an odd power of 2) are $q-1$, $q+\sqrt{2q}+1$ and $q-\sqrt{2q}+1$ all either a prime power or the product of two primes?

This happens surprisingly often. For example, $2^{11}+1 = 3 \cdot 683$ while $2^{11}-1 = 23 \cdot 89$. Are there only finitely many solutions?

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- $G = \mathrm{PSL}(2, 25)$: 325 components, each one $K_5 - P_4$.

I do not know why the components in the second and third case are the same.

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- $G = \mathrm{PSL}(3,3)$: 754 vertices, diameter 11, girth 12.
- $G = \mathrm{PSU}(3,3)$: 784 vertices, diameter 10, girth 3.

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I am sure that more computation would reveal more interesting things …

Clique number and chromatic number

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Examining these parameters for the power graphs and enhanced power graphs of groups have revealed some interesting things.

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So I put it on my blog, and a student in Ho Chi Minh City called Veronica Phan produced a short and elegant proof.
Veronica tells me she is a medical student who does mathematics as a hobby.

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In a group, the relation $x \to y$ if $y = x^n$ for some x is a partial preorder (a reflexive and transitive relation, not necessarily antisymmetric). A simple twist shows that the comparability graph of a partial preorder is perfect.

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So the power graph of a finite group is perfect, and in particular, its clique number and chromatic number are equal. But what are they?

Some number theory

We define a number-theoretic function f by the rule that $f(n)$ is the clique number of the power graph of a cyclic group of order n.

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This satisfies the recurrence

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ \phi(n) + f(n/p) & \text{otherwise,} \end{cases}$$

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From this it is easy to show that

$$\phi(n) \leq f(n) \leq c\phi(n),$$

where $c = 2.6481017597\ldots$

The clique number of the power graph

It follows that the clique number of the power graph of an arbitrary group G is the maximal value of $f(n)$, where n runs over the orders of elements of G. (This is not the same as the value of $f(n)$ where n is the maximum order, that is, the clique number of the enhanced power graph.)

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For example, let $G = \mathrm{PGL}(2, 11)$. The maximum orders of elements with respect to divisibility are 10, 11 and 12; and $f(10) = f(12) = 9$, but $f(11) = 11$. So the clique number (and chromatic number) are 11.

… for your attention.