Graphs on groups

Peter J. Cameron, University of St Andrews

Postgraduate Group Theory Conference
London, 6 July 2022

Groups and graphs

I have spent quite some time in the last couple of years thinking about various graphs defined on groups. I want to take you on a quick tour through this beautiful landscape and show you a few of the sights.

Groups and graphs

I have spent quite some time in the last couple of years thinking about various graphs defined on groups. I want to take you on a quick tour through this beautiful landscape and show you a few of the sights.

Groups are elegant; graphs are scruffy.

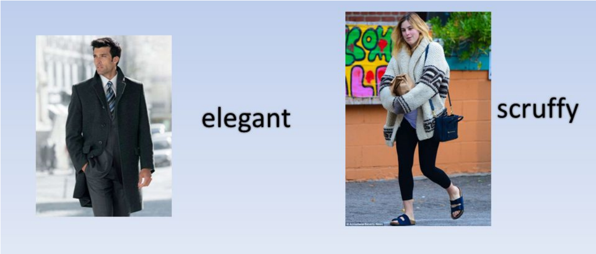
Groups and graphs

I have spent quite some time in the last couple of years thinking about various graphs defined on groups. I want to take you on a quick tour through this beautiful landscape and show you a few of the sights.
Groups are elegant; graphs are scruffy.

Groups and graphs

I have spent quite some time in the last couple of years thinking about various graphs defined on groups. I want to take you on a quick tour through this beautiful landscape and show you a few of the sights.

Groups are elegant; graphs are scruffy.



Nevertheless, they have a lot to say to one another, as we will see.

A theorem of Landau

In 1903, Landau proved the following theorem. $\kappa(G)$ is the number of conjugacy classes of the finite group G.

A theorem of Landau

In 1903, Landau proved the following theorem. $\kappa(G)$ is the number of conjugacy classes of the finite group G.

Theorem

Given a natural number k, there are only finitely many groups G for which $\kappa(G) = k$.

A theorem of Landau

In 1903, Landau proved the following theorem. $\kappa(G)$ is the number of conjugacy classes of the finite group G.

Theorem

Given a natural number k, there are only finitely many groups G for which $\kappa(G) = k$.

The proof is straightforward. Let g^G be the conjugacy class containing g, and $C_G(g)$ the centraliser of g. By the Orbit-Stabiliser Theorem, $|g^G| = |G|/|C_G(g)|$.

A theorem of Landau

In 1903, Landau proved the following theorem. $\kappa(G)$ is the number of conjugacy classes of the finite group G.

Theorem

Given a natural number k, there are only finitely many groups G for which $\kappa(G) = k$.

The proof is straightforward. Let g^G be the conjugacy class containing g, and $C_G(g)$ the centraliser of g. By the Orbit-Stabiliser Theorem, $|g^G| = |G|/|C_G(g)|$.
If $g_1, \ldots, g_k$ are conjugacy class representatives, and $|C_G(g_i)| = n_i$, then

$$|G| = \sum_{i=1}^{k} |g_i^G| = |G| \sum_{i=1}^{k} \frac{1}{n_i},$$

so $\sum_{i=1}^{k}(1/n_i) = 1$.

A theorem of Landau

In 1903, Landau proved the following theorem. $\kappa(G)$ is the number of conjugacy classes of the finite group G.

Theorem

Given a natural number k, there are only finitely many groups G for which $\kappa(G) = k$.

The proof is straightforward. Let g^G be the conjugacy class containing g, and $C_G(g)$ the centraliser of g. By the Orbit-Stabiliser Theorem, $|g^G| = |G|/|C_G(g)|$.

If $g_1, \ldots, g_k$ are conjugacy class representatives, and $|C_G(g_i)| = n_i$, then

$$|G| = \sum_{i=1}^{k} |g_i^G| = |G| \sum_{i=1}^{k} \frac{1}{n_i},$$

so $\sum_{i=1}^{k}(1/n_i) = 1$.

This equation has only finitely many solutions (the proof is an exercise!) In any given solution, the largest n_i is $|C_G(1)| = |G|$. So there are only finitely many possibilities for $|G|$.

Quantification

Landau's result implies that the minimum number $f(n)$ of conjugacy classes in a group of order n tends to infinity as $n \to \infty$. How fast?

Quantification

Landau's result implies that the minimum number $f(n)$ of conjugacy classes in a group of order n tends to infinity as $n \to \infty$. How fast?

Erdős and Turán showed that $f(n) \geq \log\log n$ (logarithms to base 2). This was improved to $\epsilon \log n/(\log\log n)^8$ by Laci Pyber; the exponent 8 was reduced to 7 by Thomas Keller, and to $3+\epsilon$ by Barbara Baumeister, Attila Maróti and Hung Tong-Viet. It is conjectured that $f(n) \geq C \log n$ holds for some constant C. In the other direction, $f(n) \leq (\log n)^3$.

Quantification

Landau's result implies that the minimum number $f(n)$ of conjugacy classes in a group of order n tends to infinity as $n \to \infty$. How fast?
Erdős and Turán showed that $f(n) \geq \log\log n$ (logarithms to base 2). This was improved to $\epsilon \log n/(\log\log n)^8$ by Laci Pyber; the exponent 8 was reduced to 7 by Thomas Keller, and to $3+\epsilon$ by Barbara Baumeister, Attila Maróti and Hung Tong-Viet. It is conjectured that $f(n) \geq C \log n$ holds for some constant C. In the other direction, $f(n) \leq (\log n)^3$.
I will show you a different kind of extension.

The SCC graph of a finite group

The soluble conjugacy class graph (for short, the SCC-graph) of G is the graph whose vertex set is the set of conjugacy classes in G, with an edge from x^G to y^G if and only if there exist $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is a soluble group.

The SCC graph of a finite group

The soluble conjugacy class graph (for short, the SCC-graph) of G is the graph whose vertex set is the set of conjugacy classes in G, with an edge from x^G to y^G if and only if there exist $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is a soluble group.
A couple of remarks:

The SCC graph of a finite group

The soluble conjugacy class graph (for short, the SCC-graph) of G is the graph whose vertex set is the set of conjugacy classes in G, with an edge from x^G to y^G if and only if there exist $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is a soluble group.
A couple of remarks:

- There are numerous variants of the definition: we could replace "soluble" by "nilpotent", "abelian", "cyclic", etc.; and there are other variants possible too.

The SCC graph of a finite group

The soluble conjugacy class graph (for short, the SCC-graph) of G is the graph whose vertex set is the set of conjugacy classes in G, with an edge from x^G to y^G if and only if there exist $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is a soluble group.
A couple of remarks:

- There are numerous variants of the definition: we could replace "soluble" by "nilpotent", "abelian", "cyclic", etc.; and there are other variants possible too.
- Sometimes we need the expanded version of this graph, where the vertex set is G, and two vertices x and y are joined if x^G and y^G are joined in the SCC-graph. (This is not the same as the solubility graph, in which x and y are joined if $\langle x, y \rangle$ is soluble; but this will come in as well.)

A theorem

So Landau bounded the order of a finite group in terms of the number of vertices of the SCC graph. We (that is, Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and I) can bound it in terms of the clique number of this graph (the size of the largest complete subgraph):

A theorem

So Landau bounded the order of a finite group in terms of the number of vertices of the SCC graph. We (that is, Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and I) can bound it in terms of the clique number of this graph (the size of the largest complete subgraph):

Theorem

Given a natural number k, there are only finitely many finite groups whose SCC graph has clique number k.

A theorem

So Landau bounded the order of a finite group in terms of the number of vertices of the SCC graph. We (that is, Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and I) can bound it in terms of the clique number of this graph (the size of the largest complete subgraph):

Theorem

Given a natural number k, there are only finitely many finite groups whose SCC graph has clique number k.

The Theorem has further consequences: for examle, given g, there are only finitely many finite groups whose SCC graph has genus g.

A theorem

So Landau bounded the order of a finite group in terms of the number of vertices of the SCC graph. We (that is, Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and I) can bound it in terms of the clique number of this graph (the size of the largest complete subgraph):

Theorem

Given a natural number k, there are only finitely many finite groups whose SCC graph has clique number k.

The Theorem has further consequences: for examle, given g, there are only finitely many finite groups whose SCC graph has genus g.
The proof requires the Classification of Finite Simple Groups. I will just give a sketch. But first, some recent results on soluble groups.

Soluble groups

If G is soluble, then clearly its solubility graph and its SCC graph are both complete.

Soluble groups

If G is soluble, then clearly its solubility graph and its SCC graph are both complete.
The converses of these results also hold. It follows from John Thompson's classification of N-groups that a finite group is soluble if and only if all its 2-generator subgroups are soluble, that is, if and only if the solubility graph is complete.

Soluble groups

If G is soluble, then clearly its solubility graph and its SCC graph are both complete.

The converses of these results also hold. It follows from John Thompson's classification of N-groups that a finite group is soluble if and only if all its 2-generator subgroups are soluble, that is, if and only if the solubility graph is complete.

Then S. Dolfi, R. M. Guralnick, M. Herzog and C. E. Praeger extended this to show that a finite group is soluble if and only if its SCC graph is complete.

Soluble groups

If G is soluble, then clearly its solubility graph and its SCC graph are both complete.

The converses of these results also hold. It follows from John Thompson's classification of N-groups that a finite group is soluble if and only if all its 2-generator subgroups are soluble, that is, if and only if the solubility graph is complete.

Then S. Dolfi, R. M. Guralnick, M. Herzog and C. E. Praeger extended this to show that a finite group is soluble if and only if its SCC graph is complete.

Moreover, the set of vertices joined to all others in the solubility graph of G is its soluble radical (largest soluble normal subgroup), a theorem of R. Guralnick, B. Kunyavskii, E. Plotkin and A. Shalev.

Soluble groups

If G is soluble, then clearly its solubility graph and its SCC graph are both complete.

The converses of these results also hold. It follows from John Thompson's classification of N-groups that a finite group is soluble if and only if all its 2-generator subgroups are soluble, that is, if and only if the solubility graph is complete.

Then S. Dolfi, R. M. Guralnick, M. Herzog and C. E. Praeger extended this to show that a finite group is soluble if and only if its SCC graph is complete.

Moreover, the set of vertices joined to all others in the solubility graph of G is its soluble radical (largest soluble normal subgroup), a theorem of R. Guralnick, B. Kunyavskii, E. Plotkin and A. Shalev.

But the analogous result for the SCC graph is false. For q a power of 2, the groups $\mathrm{PSL}(2, q)$ have one conjugacy class of involutions, and every element is inverted by some involution, so the involution class is joined to all others in the SCC graph.

Sketch proof

Step 1: We can assume that G is not soluble (by Dolfi *et al*. and Landau).

Sketch proof

Step 1: We can assume that G is not soluble (by Dolfi *et al.* and Landau).
Step 2: We can assume that the soluble radical $S(G)$ is trivial. For if $G/S(G)$ is bounded, then the number of conjugacy classes of $S(G)$ is bounded (each G-class splits into at most $|G/S(G)|$ $S(G)$-classes), so by Landau $|S(G)|$ is also bounded.

Sketch proof

Step 1: We can assume that G is not soluble (by Dolfi *et al.* and Landau).
Step 2: We can assume that the soluble radical $S(G)$ is trivial. For if $G/S(G)$ is bounded, then the number of conjugacy classes of $S(G)$ is bounded (each G-class splits into at most $|G/S(G)|$ $S(G)$-classes), so by Landau $|S(G)|$ is also bounded.
Step 3: The number of factors in the socle of G is bounded, and it suffices to assume there is just one factor.

Sketch proof

Step 1: We can assume that G is not soluble (by Dolfi *et al.* and Landau).

Step 2: We can assume that the soluble radical $S(G)$ is trivial. For if $G/S(G)$ is bounded, then the number of conjugacy classes of $S(G)$ is bounded (each G-class splits into at most $|G/S(G)|$ $S(G)$-classes), so by Landau $|S(G)|$ is also bounded.

Step 3: The number of factors in the socle of G is bounded, and it suffices to assume there is just one factor.

Step 4: Now we look through the simple groups (only a light touch is required).

Some problems

Problem

Quantify this result: that is, find a good explicit bound for $|G|$ in terms of the clique number of its SCC graph.

Problem

Does a similar theorem hold if the SCC graph is replaced by the NCC graph (the nilpotent conjugacy class graph), with g^G and h^G joined if there exist $g' \in g^G$ and $h' \in h^G$ such that $\langle g', h' \rangle$ is nilpotent), or even in the CCC graph (the commuting conjugacy class graph)?

Problem

Characterise the vertices joined to all others in the SCC graph of a group.

Other graphs

There are many further graphs associated with a group, carrying various information about the group. I am going to show you a couple of rather unrelated pieces of information.

Other graphs

There are many further graphs associated with a group, carrying various information about the group. I am going to show you a couple of rather unrelated pieces of information. Here are two further graphs. In each case the vertex set is the whole group G.

Other graphs

There are many further graphs associated with a group, carrying various information about the group. I am going to show you a couple of rather unrelated pieces of information. Here are two further graphs. In each case the vertex set is the whole group G.

- the power graph: $g \sim h$ if one of g and h is a power of the other;
- the enhanced power graph: $g \sim h$ if both g and h are powers of an element k (equivalently, $\langle g, h \rangle$ is cyclic).

Other graphs

There are many further graphs associated with a group, carrying various information about the group. I am going to show you a couple of rather unrelated pieces of information. Here are two further graphs. In each case the vertex set is the whole group G.

- the power graph: $g \sim h$ if one of g and h is a power of the other;
- the enhanced power graph: $g \sim h$ if both g and h are powers of an element k (equivalently, $\langle g, h \rangle$ is cyclic).

Clearly the edge set of the power graph is contained in that of the enhanced power graph. But maybe there is not too much difference between them …

Power graph equals enhanced power graph

The Gruenberg–Kegel graph of a finite group G (sometimes called the prime graph) has as vertices the prime divisors of $|G|$, with an edge from p to q if G contains an element of order pq.

Power graph equals enhanced power graph

The Gruenberg–Kegel graph of a finite group G (sometimes called the prime graph) has as vertices the prime divisors of $|G|$, with an edge from p to q if G contains an element of order pq. This graph was introduced by Gruenberg and Kegel to study the integral group ring of G. Their theorem, refined by later authors, describes groups for which the Gruenberg–Kegel graph is disconnected.

Power graph equals enhanced power graph

The Gruenberg–Kegel graph of a finite group G (sometimes called the prime graph) has as vertices the prime divisors of $|G|$, with an edge from p to q if G contains an element of order pq. This graph was introduced by Gruenberg and Kegel to study the integral group ring of G. Their theorem, refined by later authors, describes groups for which the Gruenberg–Kegel graph is disconnected.

Theorem

For a finite group G, the following conditions are equivalent:

Power graph equals enhanced power graph

The Gruenberg–Kegel graph of a finite group G (sometimes called the prime graph) has as vertices the prime divisors of $|G|$, with an edge from p to q if G contains an element of order pq. This graph was introduced by Gruenberg and Kegel to study the integral group ring of G. Their theorem, refined by later authors, describes groups for which the Gruenberg–Kegel graph is disconnected.

Theorem

For a finite group G, the following conditions are equivalent:

- *the power graph and enhanced power graph of G coincide;*

Power graph equals enhanced power graph

The Gruenberg–Kegel graph of a finite group G (sometimes called the prime graph) has as vertices the prime divisors of $|G|$, with an edge from p to q if G contains an element of order pq. This graph was introduced by Gruenberg and Kegel to study the integral group ring of G. Their theorem, refined by later authors, describes groups for which the Gruenberg–Kegel graph is disconnected.

Theorem

For a finite group G, the following conditions are equivalent:

- *the power graph and enhanced power graph of G coincide;*
- *every element of G has prime power order (such groups are called EPPO groups);*

Power graph equals enhanced power graph

The Gruenberg–Kegel graph of a finite group G (sometimes called the prime graph) has as vertices the prime divisors of $|G|$, with an edge from p to q if G contains an element of order pq. This graph was introduced by Gruenberg and Kegel to study the integral group ring of G. Their theorem, refined by later authors, describes groups for which the Gruenberg–Kegel graph is disconnected.

Theorem

For a finite group G, the following conditions are equivalent:

- *the power graph and enhanced power graph of G coincide;*
- *every element of G has prime power order (such groups are called EPPO groups);*
- *the Gruenberg–Kegel graph of G has no edges.*

Approximately equal?

All EPPO groups are known: the Gruenberg–Kegel theorem is an essential part of this.

Approximately equal?

All EPPO groups are known: the Gruenberg–Kegel theorem is an essential part of this.

How might we say that these graphs are approximately equal? One way is to choose a monotone graph parameter (one which cannot decrease when edges are added) and ask for which groups the values of this parameter on the two graphs are equal. Here are a couple of examples.

Approximately equal?

All EPPO groups are known: the Gruenberg–Kegel theorem is an essential part of this.

How might we say that these graphs are approximately equal?

One way is to choose a monotone graph parameter (one which cannot decrease when edges are added) and ask for which groups the values of this parameter on the two graphs are equal. Here are a couple of examples.

The matching number of a graph is the maximum number of pairwise disjoint edges.

Approximately equal?

All EPPO groups are known: the Gruenberg–Kegel theorem is an essential part of this.

How might we say that these graphs are approximately equal?

One way is to choose a monotone graph parameter (one which cannot decrease when edges are added) and ask for which groups the values of this parameter on the two graphs are equal. Here are a couple of examples.

The matching number of a graph is the maximum number of pairwise disjoint edges.

Theorem

For any finite group G, the power graph and enhanced power graph of G have the same matching number.

Clique number

Theorem

For a finite group G, the power graph and enhanced power graph have the same clique number if and only if the maximum order of an element of G is a prime power.

Simple enough, but it hides some difficulties. For example, in the group $G = \mathrm{PGL}(2, q)$, the maximum order of an element is $q + 1$; so G has this property if and only if one of

Clique number

Theorem

For a finite group G, the power graph and enhanced power graph have the same clique number if and only if the maximum order of an element of G is a prime power.

Simple enough, but it hides some difficulties. For example, in the group $G = \mathrm{PGL}(2, q)$, the maximum order of an element is $q + 1$; so G has this property if and only if one of

- q is a Mersenne prime;

Clique number

Theorem

For a finite group G, the power graph and enhanced power graph have the same clique number if and only if the maximum order of an element of G is a prime power.

Simple enough, but it hides some difficulties. For example, in the group $G = \mathrm{PGL}(2, q)$, the maximum order of an element is $q + 1$; so G has this property if and only if one of

- q is a Mersenne prime;
- $q + 1$ is a Fermat prime;

Clique number

Theorem

For a finite group G, the power graph and enhanced power graph have the same clique number if and only if the maximum order of an element of G is a prime power.

Simple enough, but it hides some difficulties. For example, in the group $G = \mathrm{PGL}(2, q)$, the maximum order of an element is $q + 1$; so G has this property if and only if one of

- q is a Mersenne prime;
- $q + 1$ is a Fermat prime;
- $q = 8$.

Clique number

Theorem

For a finite group G, the power graph and enhanced power graph have the same clique number if and only if the maximum order of an element of G is a prime power.

Simple enough, but it hides some difficulties. For example, in the group $G = \mathrm{PGL}(2, q)$, the maximum order of an element is $q + 1$; so G has this property if and only if one of

- q is a Mersenne prime;
- $q + 1$ is a Fermat prime;
- $q = 8$.

The last case uses the solution to the Catalan conjecture by Mihăilescu in 2002. Of course, the determination of Fermat and Mersenne primes is currently right out of reach!

Clique number, 2

If the clique numbers of these graphs are not equal, how far apart can they be? Let Γ_1 and Γ_2 be the power graph and enhanced power graph of G, and $\omega(\Gamma)$ the clique number of Γ.

Clique number, 2

If the clique numbers of these graphs are not equal, how far apart can they be? Let Γ_1 and Γ_2 be the power graph and enhanced power graph of G, and $\omega(\Gamma)$ the clique number of Γ. Now $\omega(\Gamma_1) \geq \phi(\omega(\Gamma_2))$, where ϕ is Euler's totient. (If G is cyclic then a clique of maximal size in Γ_1 is the set of generators of G while a clique of maximal size in Γ_2 is the whole group.)

Clique number, 2

If the clique numbers of these graphs are not equal, how far apart can they be? Let Γ_1 and Γ_2 be the power graph and enhanced power graph of G, and $\omega(\Gamma)$ the clique number of Γ. Now $\omega(\Gamma_1) \geq \phi(\omega(\Gamma_2)$, where ϕ is Euler's totient. (If G is cyclic then a clique of maximal size in Γ_1 is the set of generators of G while a clique of maximal size in Γ_2 is the whole group.)

Theorem

There is a constant c, roughly 2.6481017597*, such that*

$$\limsup \omega(\Gamma_1)/\phi(\omega(\Gamma_2)) = c.$$

Clique number, 2

If the clique numbers of these graphs are not equal, how far apart can they be? Let Γ_1 and Γ_2 be the power graph and enhanced power graph of G, and $\omega(\Gamma)$ the clique number of Γ. Now $\omega(\Gamma_1) \geq \phi(\omega(\Gamma_2)$, where ϕ is Euler's totient. (If G is cyclic then a clique of maximal size in Γ_1 is the set of generators of G while a clique of maximal size in Γ_2 is the whole group.)

Theorem

There is a constant c, roughly 2.6481017597, *such that*

$$\limsup \omega(\Gamma_1)/\phi(\omega(\Gamma_2)) = c.$$

Since $\phi(n) \geq \mathrm{e}^{-\gamma} n / \log\log n$, this says that the two parameters are not too far apart.

Clique number, 2

If the clique numbers of these graphs are not equal, how far apart can they be? Let Γ_1 and Γ_2 be the power graph and enhanced power graph of G, and $\omega(\Gamma)$ the clique number of Γ. Now $\omega(\Gamma_1) \geq \phi(\omega(\Gamma_2)$, where ϕ is Euler's totient. (If G is cyclic then a clique of maximal size in Γ_1 is the set of generators of G while a clique of maximal size in Γ_2 is the whole group.)

Theorem

There is a constant c, roughly 2.6481017597*, such that*

$$\limsup \omega(\Gamma_1)/\phi(\omega(\Gamma_2)) = c.$$

Since $\phi(n) \geq \mathrm{e}^{-\gamma} n / \log\log n$, this says that the two parameters are not too far apart.

The constant is given by $c = \sum_{k\geq 0} \prod_{i=1}^{k} \frac{1}{p_i - 1}$, where $p_1, p_2, \ldots$ are the primes in order.

Interesting graphs from groups

For my final topic, I will talk about the question of where to look to find groups giving rise to "interesting" graphs, that might be useful as networks, for example.

Interesting graphs from groups

For my final topic, I will talk about the question of where to look to find groups giving rise to “interesting” graphs, that might be useful as networks, for example.
I will talk about power graphs of simple groups.

Interesting graphs from groups

For my final topic, I will talk about the question of where to look to find groups giving rise to "interesting" graphs, that might be useful as networks, for example.
I will talk about power graphs of simple groups.
What makes a graph interesting? Perhaps we want large girth or small diameter relative to the number of vertices. Perhaps we just want a large automorphism group.

Interesting graphs from groups

For my final topic, I will talk about the question of where to look to find groups giving rise to "interesting" graphs, that might be useful as networks, for example.
I will talk about power graphs of simple groups.
What makes a graph interesting? Perhaps we want large girth or small diameter relative to the number of vertices. Perhaps we just want a large automorphism group.
So I was astonished to find that, if G is the alternating group A_5, the smallest non-abelian finite simple group, then the power graph of G has automorphism group of order 668594111536199848062615552000000.

Interesting graphs from groups

For my final topic, I will talk about the question of where to look to find groups giving rise to “interesting” graphs, that might be useful as networks, for example.
I will talk about power graphs of simple groups.
What makes a graph interesting? Perhaps we want large girth or small diameter relative to the number of vertices. Perhaps we just want a large automorphism group.
So I was astonished to find that, if G is the alternating group A_5, the smallest non-abelian finite simple group, then the power graph of G has automorphism group of order 668594111536199848062615552000000.
What is going on?

Twins

Two vertices v and w of a graph Γ are twins if they have the same neighbours, apart from possibly one another. (Sometimes, if we need to distinguish, we call them open or closed twins according as their open or closed neighbourhoods are equal.)

Twins

Two vertices v and w of a graph Γ are twins if they have the same neighbours, apart from possibly one another. (Sometimes, if we need to distinguish, we call them open or closed twins according as their open or closed neighbourhoods are equal.) If two vertices are twins, there is an automorphism of the graph which swaps these two vertices and fixes all the others. So an arbitrary graph will have a subgroup of its automorphism group consisting of a direct product of symmetric groups on the twin classes.

Twins

Two vertices v and w of a graph Γ are twins if they have the same neighbours, apart from possibly one another. (Sometimes, if we need to distinguish, we call them open or closed twins according as their open or closed neighbourhoods are equal.)

If two vertices are twins, there is an automorphism of the graph which swaps these two vertices and fixes all the others. So an arbitrary graph will have a subgroup of its automorphism group consisting of a direct product of symmetric groups on the twin classes.

Random graphs don't have twins, but graphs from groups typically do. For example, in the power graph, two elements which generate the same cyclic subgroup are twins. So, in A_5, we have a subgroup $S_4^6 \times S_2^{10}$ of such automorphisms, which are really of no interest.

Twin reduction

The process of twin reduction in a graph consists of finding a pair of twins and identifying them as long as this is possible.

Twin reduction

The process of twin reduction in a graph consists of finding a pair of twins and identifying them as long as this is possible.

Theorem

The result of twin reduction on a graph is (up to isomorphism) independent of the order in which the reduction was carried out.

I will call the result of twin reduction on a graph Γ the cokernel of Γ. So given a graph, we might want to perform twin reduction on it before looking further.

Twin reduction

The process of twin reduction in a graph consists of finding a pair of twins and identifying them as long as this is possible.

Theorem

The result of twin reduction on a graph is (up to isomorphism) independent of the order in which the reduction was carried out.

I will call the result of twin reduction on a graph Γ the cokernel of Γ. So given a graph, we might want to perform twin reduction on it before looking further.
But maybe we shrink it to a single vertex …

Cographs

A cograph is a graph containing no induced subgraph which is a 4-vertex path. This important class of graphs has been rediscovered many times.

Cographs

A cograph is a graph containing no induced subgraph which is a 4-vertex path. This important class of graphs has been rediscovered many times.

Theorem

- *A graph is a cograph if and only if its cokernel is a single vertex.*

Cographs

A cograph is a graph containing no induced subgraph which is a 4-vertex path. This important class of graphs has been rediscovered many times.

Theorem

- *A graph is a cograph if and only if its cokernel is a single vertex.*
- *Cographs form the smallest class of graphs containing the* 1*-vertex graph and closed under taking the complement or disjoint unions.*

Cographs

A cograph is a graph containing no induced subgraph which is a 4-vertex path. This important class of graphs has been rediscovered many times.

Theorem

- *A graph is a cograph if and only if its cokernel is a single vertex.*
- *Cographs form the smallest class of graphs containing the* 1*-vertex graph and closed under taking the complement or disjoint unions.*

Problem

For which groups is the power graph a cograph?

When is the power graph a cograph?

We cannot answer the question completely; but Pallabi Manna, Ranjit Mehatari and I were able to show:

When is the power graph a cograph?

We cannot answer the question completely; but Pallabi Manna, Ranjit Mehatari and I were able to show:

Theorem

The power graph of a non-abelian finite simple group G is a cograph if and only if one of the following holds:

When is the power graph a cograph?

We cannot answer the question completely; but Pallabi Manna, Ranjit Mehatari and I were able to show:

Theorem

The power graph of a non-abelian finite simple group G is a cograph if and only if one of the following holds:

- $G = \mathrm{PSL}(2, q)$ *with* q *a power of* 2*, such that each of* $q - 1$ *and* $q + 1$ *is a prime power or a product of two primes;*

When is the power graph a cograph?

We cannot answer the question completely; but Pallabi Manna, Ranjit Mehatari and I were able to show:

Theorem

The power graph of a non-abelian finite simple group G is a cograph if and only if one of the following holds:

- $G = \mathrm{PSL}(2, q)$ *with* q *a power of* 2*, such that each of* $q - 1$ *and* $q + 1$ *is a prime power or a product of two primes;*
- $G = \mathrm{PSL}(2, q)$ *with* q *an odd prime power, such that each of* $(q - 1)/2$ *and* $(q + 1)/2$ *is a prime power or a product of two primes;*

When is the power graph a cograph?

We cannot answer the question completely; but Pallabi Manna, Ranjit Mehatari and I were able to show:

Theorem

The power graph of a non-abelian finite simple group G is a cograph if and only if one of the following holds:

- $G = \mathrm{PSL}(2, q)$ *with* q *a power of* 2*, such that each of* $q - 1$ *and* $q + 1$ *is a prime power or a product of two primes;*
- $G = \mathrm{PSL}(2, q)$ *with* q *an odd prime power, such that each of* $(q - 1)/2$ *and* $(q + 1)/2$ *is a prime power or a product of two primes;*
- $G = \mathrm{Sz}(q)$*, where each of* $q - 1$*,* $q - \sqrt{2q} + 1$ *and* $q + \sqrt{2q} + 1$ *is a prime power or the product of two primes;*

When is the power graph a cograph?

We cannot answer the question completely; but Pallabi Manna, Ranjit Mehatari and I were able to show:

Theorem

The power graph of a non-abelian finite simple group G is a cograph if and only if one of the following holds:

- $G = \mathrm{PSL}(2, q)$ *with* q *a power of* 2*, such that each of* $q - 1$ *and* $q + 1$ *is a prime power or a product of two primes;*
- $G = \mathrm{PSL}(2, q)$ *with* q *an odd prime power, such that each of* $(q - 1)/2$ *and* $(q + 1)/2$ *is a prime power or a product of two primes;*
- $G = \mathrm{Sz}(q)$*, where each of* $q - 1$*,* $q - \sqrt{2q} + 1$ *and* $q + \sqrt{2q} + 1$ *is a prime power or the product of two primes;*
- $G = \mathrm{PSL}(3, 4)$.

When is the power graph a cograph?

We cannot answer the question completely; but Pallabi Manna, Ranjit Mehatari and I were able to show:

Theorem

The power graph of a non-abelian finite simple group G is a cograph if and only if one of the following holds:

- $G = \mathrm{PSL}(2, q)$ *with q a power of 2, such that each of $q-1$ and $q+1$ is a prime power or a product of two primes;*
- $G = \mathrm{PSL}(2, q)$ *with q an odd prime power, such that each of $(q-1)/2$ and $(q+1)/2$ is a prime power or a product of two primes;*
- $G = \mathrm{Sz}(q)$, *where each of $q-1$, $q-\sqrt{2q}+1$ and $q+\sqrt{2q}+1$ is a prime power or the product of two primes;*
- $G = \mathrm{PSL}(3, 4)$.

In the first three cases, deciding which values of q occur seems to be a problem beyond the current reach of number theory!

Uninteresting cases

We regard these cases as uninteresting.

Uninteresting cases

We regard these cases as uninteresting.
There is a second class of simple groups, for which the answer is only slightly more interesting. These are groups for which the cokernel of the power graph (with an isolated vertex removed if necessary) consists of many small components, all isomorphic.

Uninteresting cases

We regard these cases as uninteresting.
There is a second class of simple groups, for which the answer is only slightly more interesting. These are groups for which the cokernel of the power graph (with an isolated vertex removed if necessary) consists of many small components, all isomorphic.
Examples include:

Uninteresting cases

We regard these cases as uninteresting.
There is a second class of simple groups, for which the answer is only slightly more interesting. These are groups for which the cokernel of the power graph (with an isolated vertex removed if necessary) consists of many small components, all isomorphic.
Examples include:

- $G = A_7$: 35 components, each consisting of a tree with a trivalent centre and three arms of length 3.

Uninteresting cases

We regard these cases as uninteresting.
There is a second class of simple groups, for which the answer is only slightly more interesting. These are groups for which the cokernel of the power graph (with an isolated vertex removed if necessary) consists of many small components, all isomorphic.
Examples include:

- $G = A_7$: 35 components, each consisting of a tree with a trivalent centre and three arms of length 3.
- $G = \mathrm{PSL}(2, 23)$: 253 components, each one $K_5 - P_4$.

Uninteresting cases

We regard these cases as uninteresting.
There is a second class of simple groups, for which the answer is only slightly more interesting. These are groups for which the cokernel of the power graph (with an isolated vertex removed if necessary) consists of many small components, all isomorphic.
Examples include:

- $G = A_7$: 35 components, each consisting of a tree with a trivalent centre and three arms of length 3.
- $G = \mathrm{PSL}(2, 23)$: 253 components, each one $K_5 - P_4$.
- $G = \mathrm{PSL}(2, 25)$: 325 components, each one $K_5 - P_4$.

Uninteresting cases

We regard these cases as uninteresting.
There is a second class of simple groups, for which the answer is only slightly more interesting. These are groups for which the cokernel of the power graph (with an isolated vertex removed if necessary) consists of many small components, all isomorphic.
Examples include:

- $G = A_7$: 35 components, each consisting of a tree with a trivalent centre and three arms of length 3.
- $G = \mathrm{PSL}(2, 23)$: 253 components, each one $K_5 - P_4$.
- $G = \mathrm{PSL}(2, 25)$: 325 components, each one $K_5 - P_4$.

I do not know why the components in the second and third case are the same.

Interesting cases

However, there are several groups for which the cokernel of the power graph (minus isolated vertex) is more interesting. Here are three groups for which the graph is connected, together with the number of vertices, diameter and girth of the resulting graphs.

Interesting cases

However, there are several groups for which the cokernel of the power graph (minus isolated vertex) is more interesting. Here are three groups for which the graph is connected, together with the number of vertices, diameter and girth of the resulting graphs.

- $G = \mathrm{PSL}(3, 3)$: 754 vertices, diameter 11, girth 12.

Interesting cases

However, there are several groups for which the cokernel of the power graph (minus isolated vertex) is more interesting. Here are three groups for which the graph is connected, together with the number of vertices, diameter and girth of the resulting graphs.

- $G = \mathrm{PSL}(3,3)$: 754 vertices, diameter 11, girth 12.
- $G = \mathrm{PSU}(3,3)$: 784 vertices, diameter 10, girth 3.

Interesting cases

However, there are several groups for which the cokernel of the power graph (minus isolated vertex) is more interesting. Here are three groups for which the graph is connected, together with the number of vertices, diameter and girth of the resulting graphs.

- $G = \mathrm{PSL}(3,3)$: 754 vertices, diameter 11, girth 12.
- $G = \mathrm{PSU}(3,3)$: 784 vertices, diameter 10, girth 3.
- $G = M_{11}$: 1210 vertices, diameter 20, girth 20.

Interesting cases

However, there are several groups for which the cokernel of the power graph (minus isolated vertex) is more interesting. Here are three groups for which the graph is connected, together with the number of vertices, diameter and girth of the resulting graphs.

- $G = \mathrm{PSL}(3,3)$: 754 vertices, diameter 11, girth 12.
- $G = \mathrm{PSU}(3,3)$: 784 vertices, diameter 10, girth 3.
- $G = M_{11}$: 1210 vertices, diameter 20, girth 20.

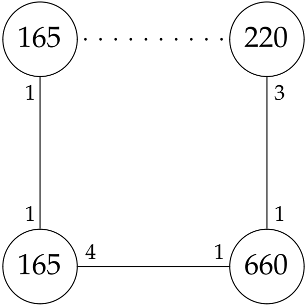
In each of these three cases, the automorphism group of the graph is equal to the automorphism group of the group.

The case $G = M_{11}$

In this case, the 1210 vertices fall into orbits of lengths 165 (twice), 220 and 660 under the action of M_{11}. The graph looks like this:

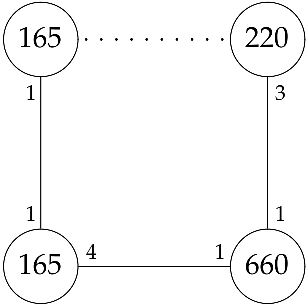
The case $G = M_{11}$

In this case, the 1210 vertices fall into orbits of lengths 165 (twice), 220 and 660 under the action of M_{11}. The graph looks like this:



The case $G = M_{11}$

In this case, the 1210 vertices fall into orbits of lengths 165 (twice), 220 and 660 under the action of M_{11}. The graph looks like this:



From this we can build a bipartite graph on $165 + 220$ vertices, where the vertices in the two parts have valencies 4 and 3 respectively.

This graph has diameter and girth 10.

This graph has diameter and girth 10.
Since it is bipartite, it is presumably the incidence graph of a nice geometry with 165 points and 220 lines, having automorphism group M_{11}. Two points lie on at most one line, and there are no triangles or quadrilaterals. I am not sure whether this geometry is already known, or what other properties it may have.

This graph has diameter and girth 10.
Since it is bipartite, it is presumably the incidence graph of a nice geometry with 165 points and 220 lines, having automorphism group M_{11}. Two points lie on at most one line, and there are no triangles or quadrilaterals. I am not sure whether this geometry is already known, or what other properties it may have.
I suspect that similar beautiful objects can be extracted from other finite simple groups in a similar way.

Some speculations

Question

For which finite simple groups is the cokernel of the power graph (less isolated vertex) connected? In particular, is this the case for most groups of Lie type with rank greater than 1*, and for most sporadic groups?*

Some speculations

Question

For which finite simple groups is the cokernel of the power graph (less isolated vertex) connected? In particular, is this the case for most groups of Lie type with rank greater than 1*, and for most sporadic groups?*

Question

If this graph is connected, is it the case that its automorphism group is the same of that of the group?

Some speculations

Question

For which finite simple groups is the cokernel of the power graph (less isolated vertex) connected? In particular, is this the case for most groups of Lie type with rank greater than 1, and for most sporadic groups?

Question

If this graph is connected, is it the case that its automorphism group is the same of that of the group?

Question

Find general results about the numbers of vertices, diameter, girth, and other parameters for the graphs in the case where they are connected.

Some speculations

Question

For which finite simple groups is the cokernel of the power graph (less isolated vertex) connected? In particular, is this the case for most groups of Lie type with rank greater than 1*, and for most sporadic groups?*

Question

If this graph is connected, is it the case that its automorphism group is the same of that of the group?

Question

Find general results about the numbers of vertices, diameter, girth, and other parameters for the graphs in the case where they are connected.

Question

What happens for other graphs defined on groups?

- Peter J. Cameron, Graphs defined on groups, *Internat. J Group Theory* **11** (2022), 43–124.
- Peter J. Cameron, Pallabi Manna and Ranjit Mehatari, On finite groups whose power graph is a cograph, *J. Algebra* **591** (2022), 59–74.
- Peter J. Cameron, V. V. Swathi and M. S. Sunitha, Matching in power graphs of finite groups, *Annals of Combinatorics* **26** (2022), 379–391.
- Parthajit Bhowal, Peter J. Cameron, Rajat Kanti Nath and Benjamin Sambale, Solvable conjugacy class graph of groups, arXiv `2112.02613`.

- Peter J. Cameron, Graphs defined on groups, *Internat. J Group Theory* **11** (2022), 43–124.
- Peter J. Cameron, Pallabi Manna and Ranjit Mehatari, On finite groups whose power graph is a cograph, *J. Algebra* **591** (2022), 59–74.
- Peter J. Cameron, V. V. Swathi and M. S. Sunitha, Matching in power graphs of finite groups, *Annals of Combinatorics* **26** (2022), 379–391.
- Parthajit Bhowal, Peter J. Cameron, Rajat Kanti Nath and Benjamin Sambale, Solvable conjugacy class graph of groups, arXiv `2112.02613`.

… for your attention.