

# Running a research discussion in South India

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The pandemic has a devastating effect on many people, who have lost loved ones or livelihoods, been afflicted with long Covid, or lost business or life opportunities.

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## History

At the British Combinatorial Conference in 2009, Shamik Ghosh posed a problem about a certain graph defined on the set of elements of a finite group. He and I worked on the problem and published a paper, and later I was able to come up with a solution. (I will tell you later what the problem was.)

## History

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## Kerala

In 2010, I was invited to an International Conference on Recent Trends in Graph Theory and Combinatorics in Kochi, Kerala, India, organised by Ambat Vijayakumar. This was a satellite meeting of the International Congress of Mathematicians held in Hyderabad that year.



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After the conference, instead of going on to the ICM, I decided to take a holiday in Kerala. (On my first visit to India, in 1988, a group of students in Mumbai had told me that the parts of India I really must see were in the far north and far south, Kashmir and Kerala.)

# God's Own Country



## A survey paper

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In the academic game, one is supposed to try to publish in the best possible journal, to score points for the department in the REF and to help one's own promotion prospects. But I am getting too old for all that, so I did something different: I put it on the arXiv, and thought that was that.

## Alireza Abdollahi

The **arXiv** is a wonderful repository, where you can upload any scientific paper, and anyone can read it for free; and the research is really up-to-the-minute. The papers are not refereed, so if you read a paper, you have to make your own estimate of its value.

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Alireza Abdollahi is editor-in-chief of the *International Journal of Group Theory*, based in Isfahan, Iran. When he saw my paper on the arXiv, he wrote to me and invited me to submit it to his journal. Since the journal is “diamond open access” (free to both authors and readers), I happily agreed. I strongly believe that we should be publishing in journals like that! It has now appeared, in volume **11** (2022), 43–124.

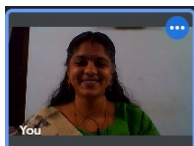


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Someone else who saw the paper on the arXiv was Ambat Vijayakumar, who as you will recall had organised the 2010 conference in Kochi. He also wrote to me and suggested running a research discussion on the topic of the paper. He and his colleague Aparna Lakshmanan would do the organisation.

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The discussion group would meet weekly, except that he invited me to give two talks in the first week to start things off. He didn't know how long it would last, maybe a few weeks; such was the interest that it ran from May until August, when I gave a summary of some of the things we had achieved, on and off line.

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So what was it about?

## The commuting graph

Back in 1955, Brauer and Fowler constructed the following graph from a group  $G$ . The vertices of the graph are the elements of  $G$ ; we put an edge joining  $x$  to  $y$  if and only if  $x$  and  $y$  commute, that is,  $xy = yx$ . So this is the **commuting graph** of  $G$ .

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They used this graph to prove that there are only a finite number of finite simple groups which contain an **involution** (an element of order 2) whose **centralizer** (the set of elements which commute with it) has a prescribed structure. This was, arguably, the first step in the thousand-mile journey to the **Classification of Finite Simple Groups**, one of the great achievements of twentieth-century mathematics.



## The power graph

As time went by, people began studying the commuting graph of a group in its own right, and found that it contained important information about the group. (For one example, a random walk on the commuting graph can be used to find a uniformly random conjugacy class of  $G$ , even though conjugacy classes can have many different sizes.)

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The question Shamik Ghosh asked me was, roughly, to what extent does the power graph determine the group? But another related question is, to what extent does the power graph determine the directed power graph? i.e. can we recover the arrows on the edges? The answer is, not completely, but we can up to isomorphism: two groups with isomorphic power graphs have isomorphic directed power graphs, even if the groups themselves are different.

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This holds only in the finite case: for infinite groups, it goes wrong, although something can be salvaged.

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Earlier, the **generating graph** of a group had been defined, with  $x$  and  $y$  joined if they generate the whole group.

## A hierarchy

What led me to write the survey was the realization that these properties form a hierarchy, where graphs later in the hierarchy are obtained from earlier graphs by adding further edges.

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It turns out that the most interesting questions are found by considering these graphs together rather than individually: rather than just work out what the power graph of a cyclic group looks like, we can ask questions such as, when can two graphs in the hierarchy be equal? If they are not equal, what can we say about the difference?

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Tamizh Chelvam pointed us to some similar graphs defined on rings rather than groups, such as zero divisor graphs. By asking questions about these similar to the ones we considered for groups, we found some new results.

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Also I continued a project with Ranjit Mehatari. There is an important class of graphs going by various names including **cographs**: this is the smallest class containing the 1-vertex graph and closed under taking complements and disjoint unions. The class of graphs for which the power graph is a cograph includes groups in which every element has prime power order (the so-called **EPPO groups**), and we are trying to determine all of these groups.

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In the **power graph**,  $x$  and  $y$  are joined if one is a power of the other. In the **enhanced power graph**, they are joined if they are both powers of an element  $z$ . A **clique** is a set of vertices any two of which are joined (that is, inducing a complete subgraph), and the **clique number** is the size of the largest clique.

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A little thought shows that a clique in the enhanced power graph of  $G$  is contained in a cyclic subgraph of  $G$ , so the clique number is the largest order of an element of  $G$ . But the power graph is a little more mysterious. However, we do know that any clique is contained in a cyclic subgroup, so it is enough to compute the clique number of the power graph of a cyclic group.

In a cyclic group of order  $n$ , there are  $\phi(n)$  elements which generate the group, where  $\phi(n)$  is **Euler's totient function**, the number of integers  $k$  with  $0 \leq k < n$  and  $\gcd(k, n) = 1$ . Any two generators are joined in the power graph, so the clique number is at least  $\phi(n)$ .

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We define a number-theoretic function  $F$  so that  $F(n)$  is the clique number of the cyclic group  $C_n$ . Then we can write a recursion for  $F$ :

$$F(1) = 1, \quad F(n) = \phi(n) + F(n/p) \text{ for } n > 1,$$

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Using this, we can show that  $F(n) < 3\phi(n)$ . In fact,

$$\limsup F(n)/\phi(n) = 2.6481017597 \dots$$

We have a formula for this mysterious constant but do not know much about it: for example, is it rational or irrational, algebraic or transcendental?

V. V. Swathi talked about the matching number of the power graph of a group (this is the maximum number of pairwise disjoint edges), and we subsequently continued this research. Although we have not yet managed to calculate the matching number for all groups, we were able to prove that the power graph and the enhanced power graph have the same matching number.

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One very nice feature of the research discussion was that, unlike some St Andrews students, the participants were very ready to contribute to the discussion with questions, additions, or remarks. This is one of the main reasons why it worked so well.

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And of course, there are plenty of interesting questions, which I am happy to suggest to anyone interested!



... for your attention.