Conjugacy class graphs on groups

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This talk will show you a couple of plums from the project.

A theorem of Landau

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Proof.

Let $x_1, \ldots, x_k$ be conjugacy class representatives. Then by the Orbit–Stabiliser Theorem, $|x_i^G| = |G|/|C_G(x_i)|$. These class sizes sum to $|G|$; so, if $n_i = |C_G(x_i)|$, we have

$$\sum_{i=1}^{k} \frac{1}{n_i} = 1.$$

This equation has only finitely many solutions [Exercise!], and in any solution $(n_1, \ldots, n_k)$, the largest n_i is $|C_G(1)| = |G|$. $\square$

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I will show you a different kind of extension.

The SCC graph of a finite group

The soluble conjugacy class graph (for short, the SCC-graph) of G is the graph whose vertex set is the set of conjugacy classes in G, with an edge from x^G to y^G if and only if there exist $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is a soluble group.

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- There are numerous variants of the definition: we could replace "soluble" by "nilpotent", "abelian", "cyclic", etc.; and there are other variants possible too.
- Sometimes we need the expanded version of this graph, where the vertex set is G, and two vertices x and y are joined if x^G and y^G are joined in the SCC-graph. (This is not the same as the solubility graph, in which x and y are joined if $\langle x, y \rangle$ is soluble; but this will come in as well.)

A theorem

So Landau bounded the order of a finite group in terms of the number of vertices of the SCC graph. We (that is, Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and I) can bound it in terms of the clique number of this graph (the size of the largest complete subgraph):

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The proof requires the Classification of Finite Simple Groups. I will just give a sketch. But first, some recent results on soluble groups.

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But the analogous result for the SCC graph is false. For q a power of 2, the groups $\mathrm{PSL}(2, q)$ have one conjugacy class of involutions, and every element is inverted by some involution, so the involution class is joined to all others in the SCC graph.

Sketch proof

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Step 2: We can assume that the soluble radical $S(G)$ is trivial. For if $G/S(G)$ is bounded, then the number of conjugacy classes of $S(G)$ is bounded (each G-class splits into at most $|G/S(G)|$ $S(G)$-classes), so by Landau $|S(G)|$ is also bounded.

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Step 3 The number of factors in the socle of G is bounded, and it suffices to assume there is just one factor.
Step 4: Now we look through the simple groups (only a light touch is required).

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- on the set of conjugacy classes, the CCC graph (two classes x^G and y^G joined if there exist $x' \in x^G$ and $y' \in y^G$ with $x'y' = y'x'$) and the PCC graph (two classes x^G and y^G joined if there exist $x' \in x^G$ and $y' \in y^G$ with one a power of the other);

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- on the whole group, the expanded CCC and PCC graphs, with x and y joined if x^G and y^G are joined in the CCC or PCC graph respectively. (These contain the commuting and power graphs respectively.)

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- *The expanded CCC graph is equal to the commuting graph if and only if G is a 2-Engel group (a group satisfying the identity $[x, y, y] = 1$).*

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- *The expanded CCC graph is equal to the commuting graph if and only if G is a 2-Engel group (a group satisfying the identity* $[x, y, y] = 1$*).*
- *The expanded PCC graph is equal to the power graph if and only if G is a Dedekind group (a group with all subgroups normal).*

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Question

Determine the groups for which the expanded SCC graph is equal to the solubility graph.

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When I wrote the abstract for this talk, I believed that the class of SCC graphs of finite groups was universal. But, embarrassingly, all I could prove was a very weak result, that every threshold graph can be embedded in the SCC graph of some group, and in order to do this, I needed a big result, the Green–Tao theorem on primes in arithmetic progression.

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Fortunately, I can do better now …

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Sketch proof:

- We show that the complete graph minus an edge can be represented: Take the first n primes $p_1, \ldots, p_n$, and take the conjugacy classes of p_i-cycles in S_N, where $N = p_{n-1} + p_n - 1$. Each pair of conjugacy classes contains commuting elements except the last two.

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- We show that the class of representable graphs on n vertices is closed under intersection of edge sets. (The SCC of a direct product of groups is the strong product of the SCCs of the two groups, and in the strong product of graphs on the same vertex set, the diagonal induces the intersection.)

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The group found in the proof can be rather large!

Other graphs

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The EPCC graphs (enhanced power conjugacy class graphs), in which x^G and y^G are joined if these classes contain elements x' and y' which generate a cyclic group, are also universal, by a similar but slightly more complicated argument.

A problem

Question

Is it true that there are only finitely many finite groups whose nilpotent conjugacy class graph (NCC graph) has given clique number? What about the commuting conjugacy class graph (CCC graph)?

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- G. Arunkumar, Peter J. Cameron, Rajat Kanti Nath and Lavanya Selvaganesh, Super graphs on groups, I, *Graphs and Combinatorics*, in press; arXiv `2112.02395`
- Parthajit Bhowal, Peter J. Cameron, Rajat Kanti Nath and Benjamin Sambale, Solvable conjugacy class graph of groups, arXiv `2112.02613`
- Peter J. Cameron, Graphs defined on groups, *Internat. J Group Theory* **11** (2022), 43–124; doi: `10.22108/ijgt.2021.127679.1681`; arXiv `2102.11177`

… for your attention.