Graphs defined on groups

Peter J. Cameron, University of St Andrews

Algebra seminar, University of York
30 September 2022

How it began

Some years ago, somebody asked me a question about this topic, which I was able to answer.

How it began

Some years ago, somebody asked me a question about this topic, which I was able to answer.

I thought no more about it until, just before the pandemic began, someone else asked me another question.

How it began

Some years ago, somebody asked me a question about this topic, which I was able to answer.
I thought no more about it until, just before the pandemic began, someone else asked me another question.
Suddenly I found myself with lots of time to sit on the sofa and think about mathematics, and this question became quite obsessive.

How it began

Some years ago, somebody asked me a question about this topic, which I was able to answer.
I thought no more about it until, just before the pandemic began, someone else asked me another question.
Suddenly I found myself with lots of time to sit on the sofa and think about mathematics, and this question became quite obsessive.
I decided that, to cure the obsession, the best thing to do was just to open a file and throw in all my thoughts, which I did.
The file grew quite long, until eventually I closed it and put it on the arXiv. (At my age I have no need to struggle to get yet another paper published in a “good” journal.)

As a result, two things happened:

As a result, two things happened:

- Alireza Abdollahi, from Isfahan in Iran, saw it and invited me to submit it to the journal of which he is editor-in-chief, the *International Journal of Group Theory*. As this is a diamond open access journal, I was happy to agree. The paper has now appeared.

As a result, two things happened:

- Alireza Abdollahi, from Isfahan in Iran, saw it and invited me to submit it to the journal of which he is editor-in-chief, the *International Journal of Group Theory*. As this is a diamond open access journal, I was happy to agree. The paper has now appeared.
- Ambat Vijayakumar, from Kochi in Kerala, India, saw it and decided to run an on-line discussion group on the topic, and invited me to help lead the discussion, which I was happy to do. The discussion group ran from March until October 2021 and was attended regularly by mathematicians from all over India and further afield.

As a result, two things happened:

- Alireza Abdollahi, from Isfahan in Iran, saw it and invited me to submit it to the journal of which he is editor-in-chief, the *International Journal of Group Theory*. As this is a diamond open access journal, I was happy to agree. The paper has now appeared.
- Ambat Vijayakumar, from Kochi in Kerala, India, saw it and decided to run an on-line discussion group on the topic, and invited me to help lead the discussion, which I was happy to do. The discussion group ran from March until October 2021 and was attended regularly by mathematicians from all over India and further afield.

The aftermath of the discussion group is that suddenly I have a large number of collaborations with Indian co-authors, and almost every day's mail brings a new manuscript for my perusal.

As a result, two things happened:

- Alireza Abdollahi, from Isfahan in Iran, saw it and invited me to submit it to the journal of which he is editor-in-chief, the *International Journal of Group Theory*. As this is a diamond open access journal, I was happy to agree. The paper has now appeared.
- Ambat Vijayakumar, from Kochi in Kerala, India, saw it and decided to run an on-line discussion group on the topic, and invited me to help lead the discussion, which I was happy to do. The discussion group ran from March until October 2021 and was attended regularly by mathematicians from all over India and further afield.

The aftermath of the discussion group is that suddenly I have a large number of collaborations with Indian co-authors, and almost every day's mail brings a new manuscript for my perusal.

I am going to tell you about some of this material.

Groups and graphs

Graphs and groups represent very contrasting parts of the mathematical universe. Groups measure symmetry; they are highly structured, elegant objects. Graphs, on the other hand, are “wild”: we can put in edges however we please. Some graphs are beautiful, but most are scruffy.

Groups and graphs

Graphs and groups represent very contrasting parts of the mathematical universe. Groups measure symmetry; they are highly structured, elegant objects. Graphs, on the other hand, are "wild": we can put in edges however we please. Some graphs are beautiful, but most are scruffy.

Groups and graphs

Graphs and groups represent very contrasting parts of the mathematical universe. Groups measure symmetry; they are highly structured, elegant objects. Graphs, on the other hand, are “wild”: we can put in edges however we please. Some graphs are beautiful, but most are scruffy.

Nevertheless, they have a lot to say to one another.

The commuting graph

I am not talking about Cayley graphs, though these are perhaps the best-known type of graphs on groups.

The commuting graph

I am not talking about Cayley graphs, though these are perhaps the best-known type of graphs on groups.

My graphs are exemplified by the commuting graph of a group, whose vertices are the group elements, two vertices x, y joined if $xy = yx$.

The commuting graph

I am not talking about Cayley graphs, though these are perhaps the best-known type of graphs on groups.

My graphs are exemplified by the commuting graph of a group, whose vertices are the group elements, two vertices x, y joined if $xy = yx$.

This graph was introduced by Brauer and Fowler in their seminal 1955 paper, where they showed that there are only finitely many finite simple groups of even order with a prescribed involution centraliser; this could be said to be the first step towards the classification of the finite simple groups.

The commuting graph

I am not talking about Cayley graphs, though these are perhaps the best-known type of graphs on groups.

My graphs are exemplified by the commuting graph of a group, whose vertices are the group elements, two vertices x, y joined if $xy = yx$.

This graph was introduced by Brauer and Fowler in their seminal 1955 paper, where they showed that there are only finitely many finite simple groups of even order with a prescribed involution centraliser; this could be said to be the first step towards the classification of the finite simple groups.

In fact Brauer and Fowler don't use the word "graph" in the paper; but their main tool is the graph distance in the induced subgraph on the non-identity elements, and the main use they make of it is to show that the diameter of this graph is surprisingly small.

A hierarchy

In order to make comparisons, we introduce several more graphs. In each case, I give the rule for joining x and y.

A hierarchy

In order to make comparisons, we introduce several more graphs. In each case, I give the rule for joining x and y.

- The power graph: one of x and y is a power of the other.

A hierarchy

In order to make comparisons, we introduce several more graphs. In each case, I give the rule for joining x and y.

- The power graph: one of x and y is a power of the other.
- The enhanced power graph: Both x and y are powers of an element z (equivalently, $\langle x, y \rangle$ is cyclic).

A hierarchy

In order to make comparisons, we introduce several more graphs. In each case, I give the rule for joining x and y.

- The power graph: one of x and y is a power of the other.
- The enhanced power graph: Both x and y are powers of an element z (equivalently, $\langle x, y \rangle$ is cyclic).
- The commuting graph, already defined (equivalently, $\langle x, y \rangle$ is abelian).

A hierarchy

In order to make comparisons, we introduce several more graphs. In each case, I give the rule for joining x and y.

- The power graph: one of x and y is a power of the other.
- The enhanced power graph: Both x and y are powers of an element z (equivalently, $\langle x, y \rangle$ is cyclic).
- The commuting graph, already defined (equivalently, $\langle x, y \rangle$ is abelian).
- The non-generating graph: $\langle x, y \rangle \neq G$.

A hierarchy

In order to make comparisons, we introduce several more graphs. In each case, I give the rule for joining x and y.

- The power graph: one of x and y is a power of the other.
- The enhanced power graph: Both x and y are powers of an element z (equivalently, $\langle x, y \rangle$ is cyclic).
- The commuting graph, already defined (equivalently, $\langle x, y \rangle$ is abelian).
- The non-generating graph: $\langle x, y \rangle \neq G$.

Each is contained in the next, except for the last two: the commuting graph is contained in the non-generating graph if and only if G is nonabelian.

A hierarchy

In order to make comparisons, we introduce several more graphs. In each case, I give the rule for joining x and y.

- The power graph: one of x and y is a power of the other.
- The enhanced power graph: Both x and y are powers of an element z (equivalently, $\langle x, y \rangle$ is cyclic).
- The commuting graph, already defined (equivalently, $\langle x, y \rangle$ is abelian).
- The non-generating graph: $\langle x, y \rangle \neq G$.

Each is contained in the next, except for the last two: the commuting graph is contained in the non-generating graph if and only if G is nonabelian.

It is clear that, above the commuting graph, we could put other graphs defined by properties imposed on $\langle x, y \rangle$: for example, nilpotence and solubility.

When are consecutive terms equal?

When we have a hierarchy it is natural to ask when two terms can be equal. For the last pair it is easy:

When are consecutive terms equal?

When we have a hierarchy it is natural to ask when two terms can be equal. For the last pair it is easy:

Proposition

Let G be a non-abelian group. Then the non-generating graph is equal to the commuting graph if and only if G is a minimal non-abelian group.

When are consecutive terms equal?

When we have a hierarchy it is natural to ask when two terms can be equal. For the last pair it is easy:

Proposition

Let G be a non-abelian group. Then the non-generating graph is equal to the commuting graph if and only if G is a minimal non-abelian group.

For if G is minimal non-abelian then any two elements which don't generate G must commute; conversely if this is true then all proper subgroups are abelian and G is minimal non-abelian.

When are consecutive terms equal?

When we have a hierarchy it is natural to ask when two terms can be equal. For the last pair it is easy:

Proposition

Let G be a non-abelian group. Then the non-generating graph is equal to the commuting graph if and only if G is a minimal non-abelian group.

For if G is minimal non-abelian then any two elements which don't generate G must commute; conversely if this is true then all proper subgroups are abelian and G is minimal non-abelian. The classification of minimal non-abelian groups was given by Miller and Moreno in 1904.

Proposition

- *The enhanced power graph of G is equal to the commuting graph if and only if G contains no subgroup $C_p \times C_p$ with p prime.*

Proposition

- *The enhanced power graph of G is equal to the commuting graph if and only if G contains no subgroup $C_p \times C_p$ with p prime.*
- *The power graph of G is equal to the enhanced power graph if and only if G contains no subgroup $C_p \times C_q$ with p, q distinct primes.*

In the first case, a theorem of Burnside shows that all Sylow subgroups of G are cyclic or generalized quaternion, and so G is determined by theorems from the “golden age” of group theory (Gorenstein–Walter and Glauberman).

Proposition

- *The enhanced power graph of G is equal to the commuting graph if and only if G contains no subgroup $C_p \times C_p$ with p prime.*
- *The power graph of G is equal to the enhanced power graph if and only if G contains no subgroup $C_p \times C_q$ with p, q distinct primes.*

In the first case, a theorem of Burnside shows that all Sylow subgroups of G are cyclic or generalized quaternion, and so G is determined by theorems from the "golden age" of group theory (Gorenstein–Walter and Glauberman).
The other case is more interesting.

The Gruenberg–Kegel graph

The Gruenberg–Kegel graph of a finite group G is the graph whose vertices are the prime divisors of G, two vertices p and q joined if G contains an element of order pq.

The Gruenberg–Kegel graph

The Gruenberg–Kegel graph of a finite group G is the graph whose vertices are the prime divisors of G, two vertices p and q joined if G contains an element of order pq.

Theorem

For the finite group G, the following conditions are equivalent:

The Gruenberg–Kegel graph

The Gruenberg–Kegel graph of a finite group G is the graph whose vertices are the prime divisors of G, two vertices p and q joined if G contains an element of order pq.

Theorem

For the finite group G, the following conditions are equivalent:

- *the power graph of G is equal to the enhanced power graph;*

The Gruenberg–Kegel graph

The Gruenberg–Kegel graph of a finite group G is the graph whose vertices are the prime divisors of G, two vertices p and q joined if G contains an element of order pq.

Theorem

For the finite group G, the following conditions are equivalent:

- *the power graph of G is equal to the enhanced power graph;*
- *the Gruenberg–Kegel graph of G has no edges;*

The Gruenberg–Kegel graph

The Gruenberg–Kegel graph of a finite group G is the graph whose vertices are the prime divisors of G, two vertices p and q joined if G contains an element of order pq.

Theorem

For the finite group G, the following conditions are equivalent:

- *the power graph of G is equal to the enhanced power graph;*
- *the Gruenberg–Kegel graph of G has no edges;*
- *every element of G has prime power order.*

The Gruenberg–Kegel graph

The Gruenberg–Kegel graph of a finite group G is the graph whose vertices are the prime divisors of G, two vertices p and q joined if G contains an element of order pq.

Theorem

For the finite group G, the following conditions are equivalent:

- *the power graph of G is equal to the enhanced power graph;*
- *the Gruenberg–Kegel graph of G has no edges;*
- *every element of G has prime power order.*

Groups satisfying the last condition are known as EPPO groups. They were first studied by Higman (who classified the soluble ones) in the 1950s. In the early 1960s, in the course of discovering his infinite family of simple groups, Suzuki found all the simple ones. The complete classification was given by Brandl in 1981, published in a rather obscure journal, which led to its rediscovery by several authors subsequently.

Differences

If two of these graphs are unequal, what can we say about their difference, the graph whose edges are those of the larger graph not in the smaller?

Differences

If two of these graphs are unequal, what can we say about their difference, the graph whose edges are those of the larger graph not in the smaller?

The difference of the non-generating and commuting graphs was studied by Saul Freedman in his PhD thesis; he concentrated mainly on questions of connectedness and diameter.

Differences

If two of these graphs are unequal, what can we say about their difference, the graph whose edges are those of the larger graph not in the smaller?

The difference of the non-generating and commuting graphs was studied by Saul Freedman in his PhD thesis; he concentrated mainly on questions of connectedness and diameter.

Recently, with Sucharita Biswas, Angsuman Das and Hiranya Kishore Dey, I have been looking at the difference of the enhanced power graph and the power graph. This is a relatively sparse graph, so when it is connected it is likely to have good properties as a network.

Differences

If two of these graphs are unequal, what can we say about their difference, the graph whose edges are those of the larger graph not in the smaller?
The difference of the non-generating and commuting graphs was studied by Saul Freedman in his PhD thesis; he concentrated mainly on questions of connectedness and diameter.
Recently, with Sucharita Biswas, Angsuman Das and Hiranya Kishore Dey, I have been looking at the difference of the enhanced power graph and the power graph. This is a relatively sparse graph, so when it is connected it is likely to have good properties as a network.
We don't have definitive results, but I will give a couple of examples. First, some remarks about useful tools.

Twin reduction

The first thing we do is to remove all the isolated vertices.

Twin reduction

The first thing we do is to remove all the isolated vertices. Two vertices of a graph are called twins if they have the same neighbours (possibly excepting one another). This is an equivalence relation on a graph, and we lose little information if we identify twins, and continue until no twins remain. The resulting graph is, up to isomorphism, independent of the reduction process.

Twin reduction

The first thing we do is to remove all the isolated vertices. Two vertices of a graph are called twins if they have the same neighbours (possibly excepting one another). This is an equivalence relation on a graph, and we lose little information if we identify twins, and continue until no twins remain. The resulting graph is, up to isomorphism, independent of the reduction process.

Many interesting classes of graphs, including perfect graphs, cographs, and chordal graphs, are determined by forbidden induced subgraphs. If the graphs in a class $\mathcal{F}$ contain no twin vertices, then a graph Γ is $\mathcal{F}$-free if and only if the result of twin reduction of Γ is $\mathcal{F}$-free.

Two examples

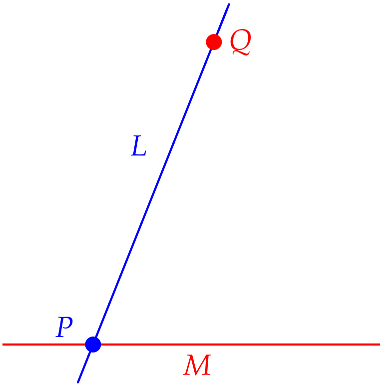
Example: The Mathieu group M_{11}.
In this case, removal of isolated vertices and twin reduction brings the number of vertices down from 7920 to 385. The resulting graph is bipartite, with bipartite sets of sizes 165 and 220, and the vertices in the two sets have valencies 4 and 3 respectively. The graph has diameter 10 and girth 10; the girth is rather large for a graph if this size. The automorphism group of the graph is just M_{11}.

Two examples

Example: The Mathieu group M_{11}.
In this case, removal of isolated vertices and twin reduction brings the number of vertices down from 7920 to 385. The resulting graph is bipartite, with bipartite sets of sizes 165 and 220, and the vertices in the two sets have valencies 4 and 3 respectively. The graph has diameter 10 and girth 10; the girth is rather large for a graph if this size. The automorphism group of the graph is just M_{11}.
Example: The group $\mathrm{PSL}(3,3)$. In this case, to our surprise, we came up with a very natural graph which has not been studied, as far as we are aware. The vertices are the ordered pairs (P, L), where P is a point and L a line of the projective plane of order 3 (so 169 vertices). The pairs fall into two types, flags (P incident with L) and antiflags (P not indident with L). The graph is bipartite: each edge joins a flag to an antiflag. Again the graph has relatively large girth, and its automorphism group is $\mathrm{Aut}(\mathrm{PSL}(3,3)$.

The rule for adjacency is that a flag (P, L) is joined to an antiflag (Q, M) if P is incident with M and Q with L.

The rule for adjacency is that a flag (P, L) is joined to an antiflag (Q, M) if P is incident with M and Q with L.



Monotone graph parameters

We feel that the power graph and enhanced graph are “not very different”. One way to substantiate this would be to take a monotone graph parameter (one which doesn’t decreases when edges are added to the graph) and compare its values on the two graphs.

Monotone graph parameters

We feel that the power graph and enhanced graph are "not very different". One way to substantiate this would be to take a monotone graph parameter (one which doesn't decreases when edges are added to the graph) and compare its values on the two graphs.

Here is an example I found with V. V. Swathi and M. S. Sunitha. The matching number of a graph is the maximum number of pairwise disjoint edges it contains. This parameter is clearly monotone.

Monotone graph parameters

We feel that the power graph and enhanced graph are "not very different". One way to substantiate this would be to take a monotone graph parameter (one which doesn't decreases when edges are added to the graph) and compare its values on the two graphs.

Here is an example I found with V. V. Swathi and M. S. Sunitha.

The matching number of a graph is the maximum number of pairwise disjoint edges it contains. This parameter is clearly monotone.

Theorem

For any finite group G, the matching numbers of the power graph and the enhancded power graph of G are equal.

Monotone graph parameters

We feel that the power graph and enhanced graph are "not very different". One way to substantiate this would be to take a monotone graph parameter (one which doesn't decreases when edges are added to the graph) and compare its values on the two graphs.
Here is an example I found with V. V. Swathi and M. S. Sunitha. The matching number of a graph is the maximum number of pairwise disjoint edges it contains. This parameter is clearly monotone.

Theorem

For any finite group G, the matching numbers of the power graph and the enhancded power graph of G are equal.

For the proof, we have to take a maximum-size matching in the enhanced power graph, carefully chosen, and for each edge not in the power graph, replace it by an edge in the power graph (with suitable readjustments elsewhere).

Extending the hierarchy into two dimensions

With G. Arunkumar, Rajat Kanti Nath and Lavanya Selvaganesh, I found a way of extending the hierarchy into a second dimension.

Extending the hierarchy into two dimensions

With G. Arunkumar, Rajat Kanti Nath and Lavanya Selvaganesh, I found a way of extending the hierarchy into a second dimension.

Let A be one of the graph types in the hierarchy, and B an equivalence relation defined on groups. We define the B superA graph of G to have vertex set G; vertices x and y are joined if there are elements x' and y', B-equivalent to x and y respectively, such that x' and y' are joined in the A-graph.

Extending the hierarchy into two dimensions

With G. Arunkumar, Rajat Kanti Nath and Lavanya Selvaganesh, I found a way of extending the hierarchy into a second dimension.
Let A be one of the graph types in the hierarchy, and B an equivalence relation defined on groups. We define the B superA graph of G to have vertex set G; vertices x and y are joined if there are elements x' and y', B-equivalent to x and y respectively, such that x' and y' are joined in the A-graph.
Now fixing A and coarsening the equivalence relation B also gives us a graph with more edges.

Extending the hierarchy into two dimensions

With G. Arunkumar, Rajat Kanti Nath and Lavanya Selvaganesh, I found a way of extending the hierarchy into a second dimension.

Let A be one of the graph types in the hierarchy, and B an equivalence relation defined on groups. We define the B superA graph of G to have vertex set G; vertices x and y are joined if there are elements x' and y', B-equivalent to x and y respectively, such that x' and y' are joined in the A-graph.

Now fixing A and coarsening the equivalence relation B also gives us a graph with more edges.

Typical equivalence relations we consider are equality (which just gives us the A graphs), conjugacy, and same order.

When are two of these equal?

A Dedekind group is a group all of whose subgroups are normal. These groups were classified by Dedekind: they are abelian groups and groups of the form $Q \times A \times B$ where Q is the quaternion group of order 8, A an elementary abelian 2-group, and B an abelian group of odd order.

When are two of these equal?

A Dedekind group is a group all of whose subgroups are normal. These groups were classified by Dedekind: they are abelian groups and groups of the form $Q \times A \times B$ where Q is the quaternion group of order 8, A an elementary abelian 2-group, and B an abelian group of odd order.

A 2-Engel group is a group satisfying the identity $[x, y, y] = 1$. A group is 2-Engel if and only if all centralisers are normal subgroups. 2-Engel groups lie between nilpotent groups of classes 2 and 3.

When are two of these equal?

A Dedekind group is a group all of whose subgroups are normal. These groups were classified by Dedekind: they are abelian groups and groups of the form $Q \times A \times B$ where Q is the quaternion group of order 8, A an elementary abelian 2-group, and B an abelian group of odd order.

A 2-Engel group is a group satisfying the identity $[x, y, y] = 1$. A group is 2-Engel if and only if all centralisers are normal subgroups. 2-Engel groups lie between nilpotent groups of classes 2 and 3.

Theorem

- *For any finite group G, the order superenhanced power graph of G is equal to the order supercommuting graph of G.*

When are two of these equal?

A Dedekind group is a group all of whose subgroups are normal. These groups were classified by Dedekind: they are abelian groups and groups of the form $Q \times A \times B$ where Q is the quaternion group of order 8, A an elementary abelian 2-group, and B an abelian group of odd order.
A 2-Engel group is a group satisfying the identity $[x, y, y] = 1$.
A group is 2-Engel if and only if all centralisers are normal subgroups. 2-Engel groups lie between nilpotent groups of classes 2 and 3.

Theorem

- *For any finite group G, the order superenhanced power graph of G is equal to the order supercommuting graph of G.*
- *The conjugacy supercommuting graph of G is equal to the commuting graph if and only if G is a 2-Engel group.*

When are two of these equal?

A Dedekind group is a group all of whose subgroups are normal. These groups were classified by Dedekind: they are abelian groups and groups of the form $Q \times A \times B$ where Q is the quaternion group of order 8, A an elementary abelian 2-group, and B an abelian group of odd order.
A 2-Engel group is a group satisfying the identity $[x, y, y] = 1$. A group is 2-Engel if and only if all centralisers are normal subgroups. 2-Engel groups lie between nilpotent groups of classes 2 and 3.

Theorem

- *For any finite group G, the order superenhanced power graph of G is equal to the order supercommuting graph of G.*
- *The conjugacy supercommuting graph of G is equal to the commuting graph if and only if G is a 2-Engel group.*
- *The conjugacy superpower graph of G is equal to the power graph if and only if G is a Dedekind group.*

Conjugacy class graphs

In a B superA graph, B-equivalent elements are twins; so it is natural to compress these graphs by shrinking each class of B to a single vertex.

Conjugacy class graphs

In a B superA graph, B-equivalent elements are twins; so it is natural to compress these graphs by shrinking each class of B to a single vertex.

For example, the commuting conjugacy class graph of G has vertex set the set of conjugacy classes in G, two classes C and D being adjacent if there exist $x \in C$ and y in D such that x and y commute (that is, $\langle x, y \rangle$ is abelian).

Conjugacy class graphs

In a B superA graph, B-equivalent elements are twins; so it is natural to compress these graphs by shrinking each class of B to a single vertex.

For example, the commuting conjugacy class graph of G has vertex set the set of conjugacy classes in G, two classes C and D being adjacent if there exist $x \in C$ and y in D such that x and y commute (that is, $\langle x, y \rangle$ is abelian).

The nilpotent conjugacy class graph and the soluble conjugacy class graph are defined analogously.

Conjugacy class graphs

In a B superA graph, B-equivalent elements are twins; so it is natural to compress these graphs by shrinking each class of B to a single vertex.

For example, the commuting conjugacy class graph of G has vertex set the set of conjugacy classes in G, two classes C and D being adjacent if there exist $x \in C$ and y in D such that x and y commute (that is, $\langle x, y \rangle$ is abelian).

The nilpotent conjugacy class graph and the soluble conjugacy class graph are defined analogously.

Using the soluble conjugacy class graph, Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and I were able to give a strengthening of a theorem of Landau from 1903.

Landau's theorem

Landau proved:

Theorem

Given a natural number k, there are only finitely many finite groups which have k conjugacy classes.

The proof is not hard. By the Orbit-Stabiliser Theorem, if a representative of the ith class has centraliser of order n_i, then the class size is $|G|/n_i$. These numbers sum to G, so we have

$$\sum_{i=1}^{k} \frac{1}{n_i} = 1.$$

This equation has only finitely many solutions.

Landau's theorem

Landau proved:

Theorem

Given a natural number k, there are only finitely many finite groups which have k conjugacy classes.

The proof is not hard. By the Orbit-Stabiliser Theorem, if a representative of the ith class has centraliser of order n_i, then the class size is $|G|/n_i$. These numbers sum to G, so we have

$$\sum_{i=1}^{k} \frac{1}{n_i} = 1.$$

This equation has only finitely many solutions.
This theorem has been studied and quantified. In particular, there is a lower bound for the number of conjugacy classes in a group of order n; the best result to date is $c \log n / (\log \log n)^{3+\epsilon}$, and it is conjectured that the correct value is $c \log n / \log \log n$.

Our theorem

Our theorem goes in a different direction. The *clique number* of a graph is the size of the largest complete subgraph. Landau's theorem says that there are only finitely many groups whose soluble conjugacy class graph has a given number of vertices. We can prove:

Our theorem

Our theorem goes in a different direction. The *clique number* of a graph is the size of the largest complete subgraph. Landau's theorem says that there are only finitely many groups whose soluble conjugacy class graph has a given number of vertices. We can prove:

Theorem

Given a positive integer k, there are only finitely many finite groups whose soluble conjugacy class graph has clique number k.

Our theorem

Our theorem goes in a different direction. The *clique number* of a graph is the size of the largest complete subgraph. Landau's theorem says that there are only finitely many groups whose soluble conjugacy class graph has a given number of vertices. We can prove:

Theorem

Given a positive integer k, there are only finitely many finite groups whose soluble conjugacy class graph has clique number k.

Unlike Landau's theorem, we need the Classification of Finite Simple Groups, and we have no decent bound for the group order in terms of k.

More on clique number

The clique number parameter is interesting for other types of graph too.

More on clique number

The clique number parameter is interesting for other types of graph too.

If a set of elements of a group has the property that any two generate a cyclic group, then the whole set is contained in a cyclic group. This means that any clique in the enhanced power graph of a group G is contained in a cyclic subgroup of G, and hence:

Proposition

The clique number of the power graph of G is the maximum order of an element of G.

More on clique number

The clique number parameter is interesting for other types of graph too.

If a set of elements of a group has the property that any two generate a cyclic group, then the whole set is contained in a cyclic group. This means that any clique in the enhanced power graph of a group G is contained in a cyclic subgroup of G, and hence:

Proposition

The clique number of the power graph of G is the maximum order of an element of G.

Since the power graph is contained in the enhanced power graph, we see that any clique in the power graph is contained in a cyclic subgroup, so:

Proposition

The clique number of the enhanced power graph of G is equal to the largest clique number of any cyclic subgroup of G.

For cyclic groups

I looked at this with Ajay Kumar, Lavanya Selvaganesh and T. Tamizh Chelvam.

For cyclic groups

I looked at this with Ajay Kumar, Lavanya Selvaganesh and T. Tamizh Chelvam.

Define the number-theoretic function $f(n)$ to be the clique number of the cyclic group of order n.

For cyclic groups

I looked at this with Ajay Kumar, Lavanya Selvaganesh and T. Tamizh Chelvam.
Define the number-theoretic function $f(n)$ to be the clique number of the cyclic group of order n.
Since the generators of C_n form a clique in the power graph, we see that $f(n) \geq \phi(n)$, where ϕ is Euler's totient function.

For cyclic groups

I looked at this with Ajay Kumar, Lavanya Selvaganesh and T. Tamizh Chelvam.

Define the number-theoretic function $f(n)$ to be the clique number of the cyclic group of order n.

Since the generators of C_n form a clique in the power graph, we see that $f(n) \geq \phi(n)$, where ϕ is Euler's totient function.

Since $\phi(n) \geq e^{-\gamma} n / \log\log n$, we conclude that the clique number of the enhanced power graph is only a little greater than the clique nunber of the power graph: more evidence that, for any group, these graphs are not very different.

For cyclic groups

I looked at this with Ajay Kumar, Lavanya Selvaganesh and T. Tamizh Chelvam.

Define the number-theoretic function $f(n)$ to be the clique number of the cyclic group of order n.

Since the generators of C_n form a clique in the power graph, we see that $f(n) \geq \phi(n)$, where ϕ is Euler's totient function.

Since $\phi(n) \geq \mathrm{e}^{-\gamma} n / \log\log n$, we conclude that the clique number of the enhanced power graph is only a little greater than the clique nunber of the power graph: more evidence that, for any group, these graphs are not very different.

But we can say more …

Theorem

- $f(1) = 1$, *and* $f(n) = \phi(n) + f(n/p)$ *for* $n > 1$, *where* p *is the smallest prime divisor of* n.
- *There is a constant* $c = 2.6481017597\ldots$ *with the property that* $f(n)/\phi(n) \leq c$.

The limit superior of the ration $f(n)/\phi(n)$ is given by the formula

$$c = \sum_{i \geq 0} \prod_{j=1}^{i} \frac{1}{p_j - 1}.$$

This sum converges very rapidly so it is easy to find good approximations for c. But is it algebraic or transcendental?

Summary

So this has been an exciting time for me. In the last two years I have twelve papers on this topic published or in press, and nearly as many more in preparation, with a wide range of coauthors, mostly from India but also from the UK, Germany, Iran, Vietnam and Australia.

Summary

So this has been an exciting time for me. In the last two years I have twelve papers on this topic published or in press, and nearly as many more in preparation, with a wide range of coauthors, mostly from India but also from the UK, Germany, Iran, Vietnam and Australia.

A final note for semigroup theorists: these graphs have not been much studied for semigroups, though (for example) the power graph was first defined for these. For sure there are interesting things to find!

Summary

So this has been an exciting time for me. In the last two years I have twelve papers on this topic published or in press, and nearly as many more in preparation, with a wide range of coauthors, mostly from India but also from the UK, Germany, Iran, Vietnam and Australia.

A final note for semigroup theorists: these graphs have not been much studied for semigroups, though (for example) the power graph was first defined for these. For sure there are interesting things to find!

So there is far more to tell, but this is not the time to tell it …

Summary

So this has been an exciting time for me. In the last two years I have twelve papers on this topic published or in press, and nearly as many more in preparation, with a wide range of coauthors, mostly from India but also from the UK, Germany, Iran, Vietnam and Australia.

A final note for semigroup theorists: these graphs have not been much studied for semigroups, though (for example) the power graph was first defined for these. For sure there are interesting things to find!

So there is far more to tell, but this is not the time to tell it …

… for your attention.