Super graphs on groups

Peter J. Cameron, University of St Andrews

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Graphs on groups

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The simplest example is the commuting graph of G, in which x and y are joined if and only if $xy = yx$. This was used by Brauer and Fowler in their seminal 1955 paper on centralizers of involutions in simple groups of even order. (Brauer and Fowler don't use the word "graph", but make extensive use of the distance function in the graph after the identity is deleted.

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In the commuting graph, elements of the centre are joined to everything; so for considerations of connectedness, diameter, etc., it makes sense to remove the centre. But for other questions it is better to leave it in, and I shall do so. I will denote this graph by $\mathrm{Com}(G)$.

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$E(\mathrm{Pow}(G)) \subseteq E(\mathrm{EPow}(G)) \subseteq E(\mathrm{Com}(G))$.

There are several further graphs which can be defined, including the solubility graph (with an edge joining x and y if $\langle x, y \rangle$ is soluble) and the generating graph (with an edge if $\langle x, y \rangle = G$). But I will stick with three for now.

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See my paper in the *International Journal of Group Theory*, **11** (2022), 43–124.

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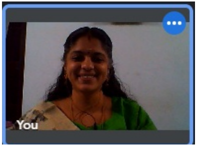

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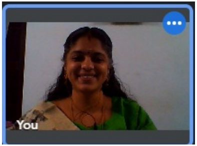

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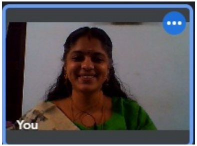

One of the participants was Lavanya Selvaganesh, who defined a graph she called the superpower graph of G, in which x and y are joined if there exist x' and y', having the same order as x and y respectively, which are joined in the power graph. This led us (G. Arunkumar, Rajat Kanti Singh, Lavanya Selvaganesh and me) to the following generalization (arXiv `2112.02395`) ...

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- The B super$A$ graph on G has vertex set G, with x joined to y if there exist elements x' and y', B-equivalent to x and y respectively, such that x' and y' are A-adjacent. (By convention, we take vertices in the same B-class to be adjacent.)

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A condensed supergraph can be expanded by blowing each vertex up to a clique of the appropriate size.

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With equality, we just get the original graph A. I will denote the conjugacy and order superpower graphs by $\mathrm{Conj\_Pow}(G)$ and $\mathrm{Ord\_Pow}(G)$, with similar notation for the superenhanced power graphs and supercommuting graphs.

The 2-dimensional hierarchy

Here is the resulting 2-dimensional hierarchy:

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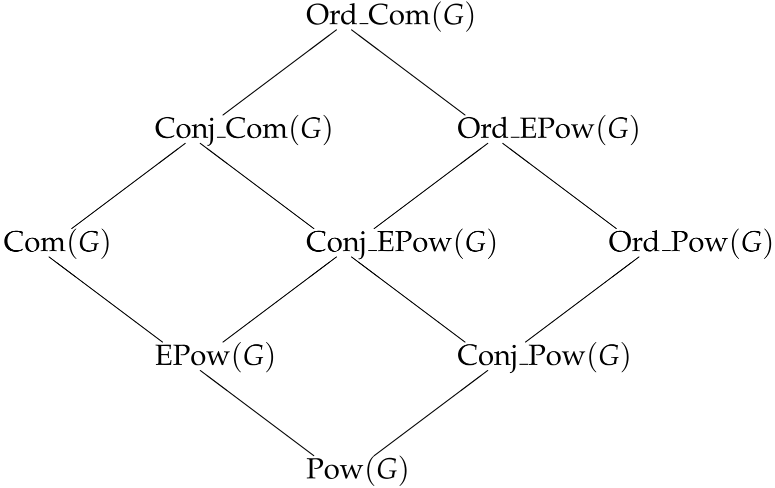

Motivation

To convince you that this is not just generalization for its own sake, I need to show you two things: the supergraphs are closely connected to the group structure; and there are some interesting results, or results with interesting proofs, concerning these graphs. This I hope to do in the rest of this talk.

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The first result I will show you was done as a "proof of concept".

Completeness

Theorem

The following table describes groups whose power graph, enhanced power graph, commuting graph, or their conjugacy or order supergraph is complete.

	power graph	enhanced power graph	commuting graph
equality	cyclic p-group	cyclic	abelian
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order	*p-group*	$(*)$	$(*)$

Here $(*)$ means that the group G has an element whose order is the exponent m of G; equivalently, the *spectrum* of G (the set of orders of elements of G) is the set of all divisors of m.

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- *The conjugacy supercommuting graph of G is equal to the commuting graph if and only if G is a 2-Engel group, that is, satisfies the identity* $[x, y, y] = 1$*;*

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Theorem

- *The conjugacy supercommuting graph of G is equal to the commuting graph if and only if G is a 2-Engel group, that is, satisfies the identity* $[x, y, y] = 1$*;*
- *the conjugacy superpower graph of G is equal to the power graph if and only if G is a Dedekind group, that is, one in which every subgroup is normal.*

Comments

Dedekind groups are all known. Such a group is either abelian or of the form $A \times B \times C$ where A is a quaternion group, B an elementary abelian 2-group, and C an abelian group of odd order.

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The only proof we found in the literature was a `StackExchange` post by Korhonen, using a result of Kappe. Information on earlier proofs welcome!

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There are twelve edges in the diagram I showed you earlier, and we have only dealt with five of them, so plenty remains to be done.

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- *The set of dominant vertices in the commuting graph is the centre of G.*

And for the super graphs . . .

Theorem

- *If A is the power graph, enhanced power graph, or commuting graph, then the set of dominant vertices in the conjugacy super$A$ graph of G is the same as the set of dominant vertices in the A graph.*

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- *Let G be a group not or prime power order, having exponent m. Then the set of dominant vertices in the order superpower graph consists of the identity and the elements of order m (if any).*

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The remaining case is the order supercommuting graph, which is an open problem.

Moving up

Some of the condensed supergraphs had been looked at earlier. We move up in the hierarchy and examine the partition into conjugacy classes. Thus the condensed conjugacy super$A$ graph has vertices the conjugacy classes, two classes C and D adjacent if there are elements $x \in C$, $y \in D$ which are adjacent in the graph A.

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The rule for adjacency in the commuting graph can be written: x and y are joined if $\langle x, y \rangle$ is abelian. Inspired by this one can define the nilpotence and solubility graphs, in which x and y are joined if the group they generate is nilpotent or soluble respectively.

The SCC-graph

The condensed conjugacy supergraphs for the commuting and nilpotent graphs were studied by Herzog, Longobardi and Maj and by Mohammadian and Erfanian respectively, under the names commuting conjugacy class graph (CCC-graph) and nilpotent conjugacy class graph (NCC-graph) respectively. We examined the analogous soluble conjugacy class graph (SCC-graph). Recall that the vertices are the conjugacy classes, two classes C and C' joined if there exist $x \in C$ and $y \in C'$ such that $\langle x, y \rangle$ is soluble.

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The team grew as the research progressed. The paper is by Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and me (arXiv `2112.02613`).

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Now if G is a soluble group, then any two elements generate a soluble group, and so the soluble graph is complete. The converse is also true. It follows from Thompson's classification of N-groups that, if any two elements of G generate a soluble group, then G is soluble. This was generalized by Dolfi, Guralnick, Herzog and Praeger as follows:

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Combining this with Landau's result, we see that there are only finitely many finite soluble groups whose SCC-graph has given clique number.

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In particular, the finite groups in which the clique number of the graph is at most 3 are the cyclic groups of orders 1, 2 and 3 and the symmetric group of degree 3.
We have not examined the growth rate for the number or largest order of such a group. Also, we have not tried to find all groups whose SCC-graph has clique number 4 (these include the alternating group A_5).

Proof sketch

The proof proceeds as follows. By the results on the previous slide we may assume that G is not soluble. Then we can reduce to the case where the soluble radical is trivial, and further to the case where the socle is almost simple; then the Classification of Finite Simple Groups gives the result.

Proof sketch

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An alternative proof when we know the group is simple is the following. If the clique number of the SCC-graph is bounded, then the number of prime divisors of an element order in G is bounded. A recent result of Hung and Yang then bounds the number of prime divisors of G. Then we can bound these prime divisors, and hence the exponent of G. But there are only finitely many finite simple groups of given exponent (an old result of Gareth Jones).

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The enhanced power graph and commuting graph are both universal. Indeed the following theorem holds:

Theorem

Given a finite complete graph with edges coloured red, green and blue in any manner, there is a group G such that red edges belong to $\mathrm{EPow}(G)$*, green edges to* $\mathrm{Com}(G)$ *but not* $\mathrm{EPow}(G)$*, and blue edges not to* $\mathrm{Com}(G)$*.*

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This theorem shows that enhanced power graphs are universal (ignoring the blue-green distinction), and commuting graphs are universal (ignoring the red-green distinction), but also graphs which are the differences between the edge sets of these two are universal (ignoring the red-blue distinction).

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They are also the class of graphs obtained by adding vertices one at a time, each new vertex joined to either all or none of the existing vertices.

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SCC-graphs

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Certainly not all SCC-graphs are threshold. Taking $G = S_7$, we can use the classes of a 7-cycle, two disjoint 3-cycles, a 4-cycle, and a 5-cycle to get a path on four vertices.

Sketch proof

Here is a brief sketch. We are given a threshold graph, with vertex weights and threshold. We adjust these slightly so that they are rational, and multiply up to make them integers.

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Now we apply Green and Tao to scale up further so that the weights are primes chosen from an arithmetic progression, while the threshold is greater than all the weights.
Now let G be the symmetric group whose degree is the threshold, and take the vertices to be conjugacy classes of cycles of the appropriate prime length. If the sum of two primes is below the threshold, there are conjugates with disjoint support, generating an abelian group; but if it is above, then any two supports intersect, and so the prime cycles generate the alternating group.

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… for your attention.