Synchronization: from automata to weakly perfect graphs

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Reset words are useful to bring a machine into a known state before applying further transformations to it.

An infamous problem

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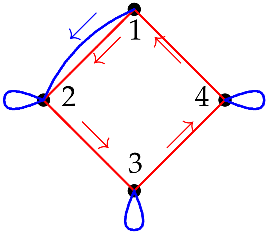

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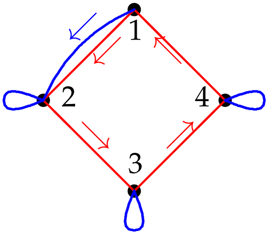

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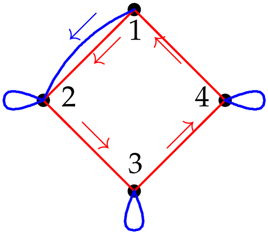

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So the Černý conjecture is a question about transformation monoids, and semigroups enter the picture.

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The endomorphisms of a graph form a transformation monoid. Moreover, as long as the graph has at least one edge, its endomorphism monoid is not synchronizing, since that edge cannot be collapsed by any endomorphism.

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If a graph is weakly perfect, then it admits an endomorphism carrying each colour class in a minimal colouring to a vertex in a maximal clique.

Synchronization and endomorphisms

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Now we have a pleasant surprise:

Theorem

A transformation monoid M is non-synchronizing if and only if there is a non-trivial graph Γ on the domain such that M is contained in the endomorphism monoid of Γ. Moreover, we can assume that Γ is weakly perfect.

Sketch proof

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The first point is clear; I will outline the second. If it fails, then some element $m \in M$ maps an edge $\{s, t\}$ to either a single vertex or a non-edge. The first case contradicts the definition; in the second case, there is $m' \in M$ with $(sm)m' = (tm)m'$, so mm' maps s and t to the same place.

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For the last point, take an element $m \in M$ of minimal rank; then m is a colouring of the graph and its image is a clique.

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However, the advantage is that we can potentially show that whole classes of automata are synchronizing, or non-synchronizing.

In our introductory example, one of the basic transitions of the automaton was a permutation (generating a cyclic group of order 4), while the other was not. We now turn to automata with the property that all but one of their transitions are permutations.

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Let Ω be a set. I will call a structure on Ω trivial if it is invariant under the symmetric group, the group of all permutations of Ω. Many important permutation group properties can be defined saying that a permutation group G on Ω (a subgroup of $\mathrm{Sym}(\Omega)$) has property P if it preserves no non-trivial structure of type X on Ω.

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Now we can add one further property:

- A permutation group G on Ω is synchronizing if it preserves no no-trivial weakly perfect graph on Ω.

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Conversely, if there exists f such that $\langle G, f \rangle$ is not synchronizing, then this monoid is contained in $\mathrm{End}(\Gamma)$, where Γ is a non-trivial graph with clique number equal to chromatic number; clearly $G \leq \mathrm{Aut}(\Gamma)$.

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- A synchronizing group is primitive. For if G is transitive and preserves a non-trivial partition P of Ω, then all parts of P have the same size, and the disjoint union of complete graphs on the parts of P is G-invariant and weakly perfect.

The O'Nan–Scott Theorem

The structure of finite primitive permutation groups is given by this theorem, which was proved independently by Michael O'Nan and Leonard Scott in 1979. However, much of the theorem, including what we need, was in Camille Jordan's *Traité des Substitutions* a hundred years earlier. The groups in the theorem will be explained on the next few slides.

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- *G is almost simple.*

Wreath product, affine and almost simple groups

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If G is a non-abelian simple group, we have a simple diagonal group; these are the groups in the O'Nan–Scott theorem.

However, the construction of these groups does not require G to be simple, or even finite.

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The most succinct description of them is as follows. Let m be an integer at least 2, and G a group, finite or infinite. The diagonal graph $\Gamma_D(G,m)$ is the Cayley graph $\mathrm{Cay}(G^m, \bigcup S_i)$ with

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When $m = 2$, this is the strongly regular Latin square graph associated with the Cayley table of G. When $|G| = 2$, it is the distance-transitive folded cube. We think these graphs could be of wider interest to algebraic graph theorists.

Why these groups are non-synchronzing

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In fact, except for a few small cases, $\Gamma_D(G, m)$ has clique number $|G|$ and, if m or $|G|$ is odd or G has non-cyclic Sylow 2-subgroups, it also has chromatic number $|G|$. We conjecture that in the remaining case the chromatic number is $|G| + 2$.

Why these groups are non-synchronzing

Based on the proof in 2009 of the Hall–Paige conjecture, it is possible to show that a diagonal graph over a finite simple group has clique number equal to chromatic number. Hence permutation groups of simple diagonal type with dimension at least 2 are non-synchronizing.
In fact, except for a few small cases, $\Gamma_D(G, m)$ has clique number $|G|$ and, if m or $|G|$ is odd or G has non-cyclic Sylow 2-subgroups, it also has chromatic number $|G|$. We conjecture that in the remaining case the chromatic number is $|G| + 2$.

There remain the case $m = 1$. These contain the group $G \times G$, acting on G by left and right multiplication, together with inversion and automorphisms of G. A recent result of John Bamberg, Michael Giudici, Jesse Lansdown and Gordon Royle shows that these groups may or may not be synchronizing.

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