Complete mappings of semigroups

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(joint with João Araújo, Wolfram Bentz and Michael Kinyon)

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Introduction

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Everything in this talk will be finite.

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The very general question, which I am not going to address, is:

Question

Which magmas have complete mappings?

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By similar reasoning, and using Burnside's transfer theorem, they proved

Theorem

Let G be a group of even order whose Sylow 2-subgroups are cyclic. Then G has no complete mapping.

The Hall–Paige conjecture

Hall and Paige conjectured the converse:

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If either $|G|$ is odd or G has non-cyclic Sylow 2-subgroups, then G has a complete mapping.

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So the conjecture was proved, though Bray's proof was not published until 2020.

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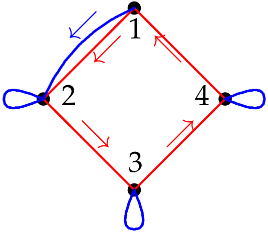

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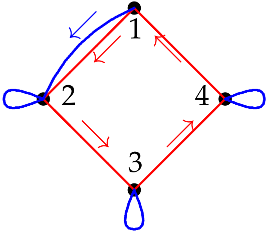

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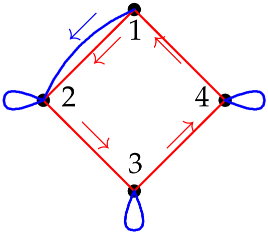

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Show that, if an n-state automaton is synchronizing, it has a reset word of length at most $(n-1)^2$.

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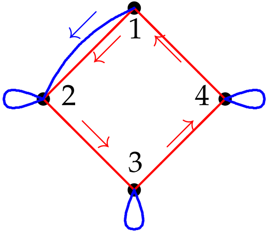

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Problem

Show that, if an n-state automaton is synchronizing, it has a reset word of length at most $(n-1)^2$.

This is the Černý conjecture, posed in the 1960s and still open.

Automata to transformation semigroups

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So an automaton gives rise to a transformation monoid on Ω with a distinguished set of generators (corresponding to the letters in the alphabet).

The automaton is synchronizing if and only if the monoid contains an element of rank 1. So we call a transformation monoid synchronizing if it contains a rank 1 transformation.

Synchronizing permutation groups

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I do not intend to give a course on permutation groups here, but in the next slide I will summarise our current knowledge.

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For two-factor socles, the groups may or may not be synchronizing.

Further developments

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But my goal is to speak about semigroups …

A reduction

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Now let J_a denote the $\mathcal{J}$-class of a, and add a new element 0 to J_a, with a multiplication defined on this set J_a^0 by

$$u \times v = \begin{cases} uv & \text{if } u, v, uv \in J_a, \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem

S has a complete mapping if and only if J_a^0 has a complete mapping for all $a \in S$.

Rees 0-matrix semigroups

Let G be a group and 0 an element not in G. Let I and Λ be finite index sets, and let P be a $I \times \Lambda$ matrix with entries from $G \cup \{0\}$. The Rees 0-matrix semigroup with sandwich matrix P is the set

$$\mathcal{M}^0[G, I, \Lambda; P] = (I \times G \times \Lambda) \cup \{0\}$$

with 0 a zero element and multiplication of other elements given by

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, g p_{j,\lambda} h, \mu) & \text{if } p_{j,\lambda} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem (Rees)

Let S be a semigroup and $a \in S$. Then either J_a^0 satisfies $xy = 0$ for all x, y, or J_a^0 is a Rees 0-matrix semigroup.

This reduces our problem to dealing with Rees 0-matrix semigroups.

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I don't know how other semigroup theorists feel about this. I have always felt that Sylow's Theorem is the analogous test for someone to be a group theorist.

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Note that the first two conditions are necessary and sufficient for G to have a complete mapping.

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For the third we give a direct construction.
For the fourth, we have to partition the sandwich matrix into pieces and handle them separately; in one case, the pieces are not rectangular, so we have to find a complete mapping in a "partial semigroup".
For the converse, we may suppose that G has no complete mapping, so that its Sylow 2-subgroups are cyclic; an easy reduction allows us to assume that G is a cyclic 2-group. Then arguments similar to the one I gave earlier for C_4 apply.

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Here is what we think holds. The pattern of a sandwich matrix is the matrix obtained by replacing the non-zero elements by 1; it can be regarded as a sandwich matrix over the trivial group.

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We have shown that

$$(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d),$$

and that if $|I| = |\Lambda|$ then all four are equivalent.

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So probably the thing we are missing to complete the proof is a combinatorial argument . . .

Further results

We have some apparently unrelated general results. For example, a short and ingenious argument involving simple manipulations shows:

Theorem

A semigroup which has a complete mapping is regular.

But we have not been able to use this or related results to help answer our question for Rees 0-matrix semigroups, since these are regular . . .

… for your attention.