The ADE affair

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University of St Andrews

UBI, Covilhã
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Calling card

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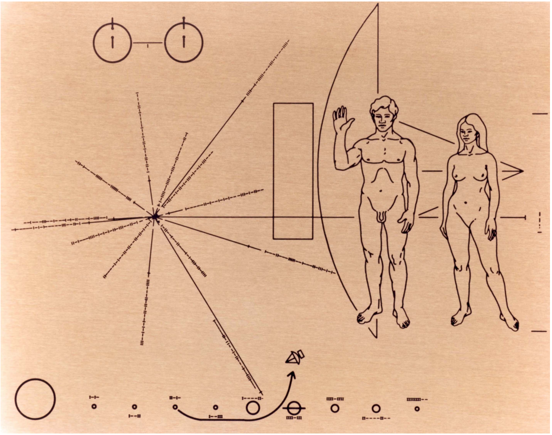
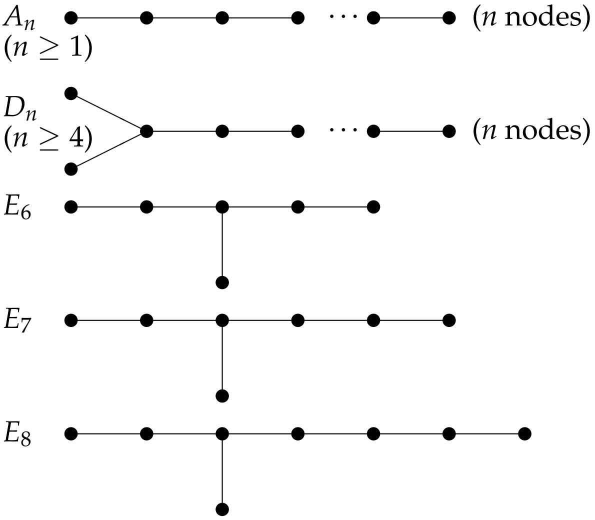
The Pioneer engineers decided that this would do:



Francis Buekenhout had another idea:



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- … the Coxeter groups generated by reflections, or of Weyl groups with roots of equal length.

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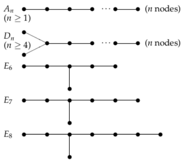
I will say a bit about some of this rich tapestry.

The extended diagrams



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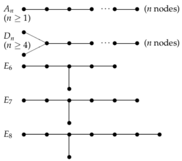

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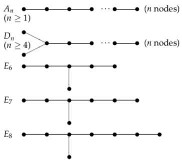

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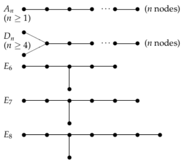

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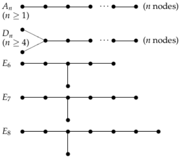
- for A_n, the new vertex is joined to both ends of the path, forming an $(n+1)$-cycle;
- for D_n, it creates a fork at the other end of the diagram, or a $K_{1,4}$ if $n = 4$;
- for E_n, it extends one of the arms, so that the numbers of vertices on the arms are $(3,3,3)$ ($n = 6$), $(2,4,4)$ ($n = 7$), or $(2,3,6)$ ($n = 8$).

What are these diagrams?



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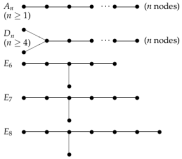

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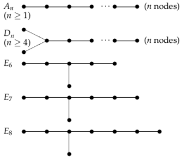

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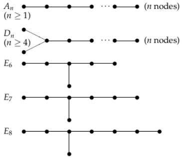

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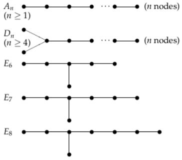

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Although this theorem was proved in 1969, it is in some sense implicit in the classification of simple Lie algebras over $\mathbb{C}$.

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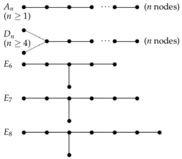
In particular, it does not contain $\tilde{A}_n$ (a cycle), and so is a tree; it does not contain $\tilde{D}_n$, and so has at most one branchpoint, such a point having valency at most 3; and it does not contain $\tilde{E}_n$, and so the lengths of the three arms are restricted to the appropriate values.

Polyhedra and tessellations



The numbers of vertices on the arms of the E_n diagrams are $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ for $n = 6, 7, 8$ respectively. This should remind you of the regular polyhedra in 3-space.

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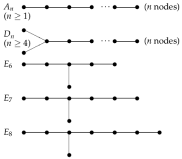

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The corresponding numbers for the extended diagrams are, as we saw, $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$ respectively, corresponding to the regular tessellations of the Euclidean plane by triangles, squares and hexagons.

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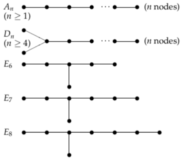

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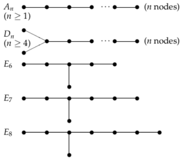
These are not accidental; we will return to them later.

A detour



By the Perron–Frobenius Theorem, a connected graph has a unique eigenvector (up to scalar multiple) corresponding to the greatest eigenvalue; this eigenvector has all its entries positive.

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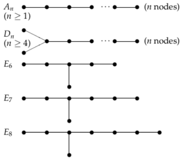

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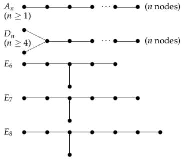

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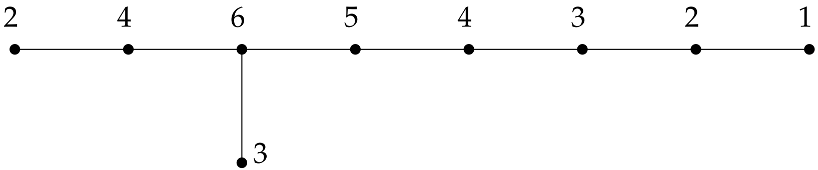
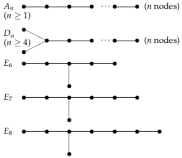
It is a simple exercise to write down such a labelling. See the next slide for $\tilde{E}_8$.

An eigenvector



Finite rotation groups



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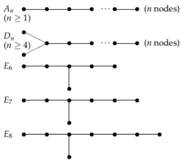

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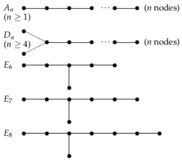

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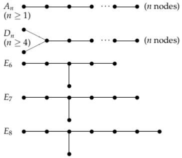

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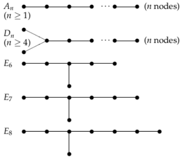

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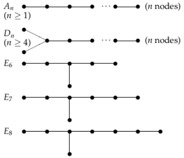

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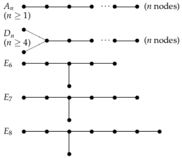

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The correspondence with the ADE diagrams is clear.

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Each of these groups comes with a "natural" two-dimensional unitary representation ρ, which is in fact self-dual (implying that $\rho \otimes \rho$ contains the trivial representation).

McKay's observation

From the preceding, we see that there is a graph structure on the set of irreducible complex representations of $\tilde{G}$: an edge joins the representations σ_1 and σ_2 if σ_2 is a constituent of the representation $\sigma_1 \otimes \rho$.

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So we have a positive eigenvector with eigenvalue 2.
We conclude that the graph is an extended ADE diagram.
Indeed, the correspondence agrees with the one we just observed for the rotation groups.

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I am afraid I am not competent to tell you about this, but I am very happy to have had a part in the story!

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The next day, we visited Jean-Marie in Brussels, who explained that he had also succeeded, and found one infinite family and three sporadic examples.

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The moral of the ADE classification is that, if you solve a classification problem to which the answer turns out to be "two infinite familes and three sporadic examples", then chances are that your problem is connected with the ADE affair.

In the 1970s, two of my collaborators, Jaap Seidel and Jean-Marie Goethals, were working on graphs with smallest eigenvalue -2.

I visited Jaap in Eindhoven; he explained to me that he had solved a technical problem that arose, and found one infinite family and three sporadic examples.

The next day, we visited Jean-Marie in Brussels, who explained that he had also succeeded, and found one infinite family and three sporadic examples.

On comparing their work, each of them had missed a different infinite family. So it was clearly an ADE problem, as I will now describe.

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- $\sigma_v(w) = w - 2(v \cdot w)/(v \cdot v)v$, an integer linear combination of v and w. From this, it can be shown that R spans a lattice (a root lattice).

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It can be shown that there is a basis consisting of roots with non-positive inner products. If G is the Gram matrix, then (after normalisation) we have $G = 2I - A$, where A is the adjacency matrix of a graph. Since G is positive definite, A has greatest eigenvalue less than 2.

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Thus, if the root system is indecomposable (that is, the graph is connected), it is an ADE diagram. Moreover, the graph determines the root system.

From the icosahedron to E_8

Is there a direct connection between the 3-dimensional rotation groups and the root systems, underlying the McKay correspondence? A beautiful construction was found recently by Pierre Dechant, using Clifford algebras; but I cannot describe it here. I merely note one small piece of numerology: the number of roots in the E_8 root system (240) is twice the number of rotations and reflections of the icosahedron.

Star-closed sets

Let A be the adjacency matrix of a graph with least eigenvalue -2. Then $A+2I$ is positive semidefinite, and so is the matrix of inner products of a set of vectors in $\mathbb{R}^n$, where n is the comultiplicity of -2 as an eigenvalue. Clearly any two of these vectors make an angle 90° or 60°.

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Now a geometric argument shows that a set of lines through the origin in $\mathbb{R}^n$, in which any two lines make angle 90° or 60°, and is maximal with respect to this property, is star-closed: that is, if two lines in the set make angle 60°, then the third line in their plane at 60° to both is also in the set.

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Taking vectors of fixed length on the lines in such a star-closed set, we obtain a root system!

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Theorem

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The paper containing this theorem (with Goethals, Seidel, and Ernest Shult) is one of my most cited.

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