Graphs defined on groups

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- We learn new results about groups.
- Using graphs we can characterise some important classes of groups.
- We might find some beautiful graphs in the process.

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- The generating graph: $x \sim y$ if $\langle x, y \rangle = G$.
- The power graph: $x \sim y$ if one of x and y is a power of the other.

The commuting graph

The first appearance of the commuting graph was in a paper of Brauer and Fowler in 1955. Curiously they did not use the word "graph" anywhere in the paper; but central to their argument is the graph distance in this graph on a finite simple group, with the identity removed (since it commutes with everything).

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This very important paper is arguably the first step on the road to the Classification of Finite Simple Groups (though that took 50 years to complete). An involution is an element of order 2; its centraliser is the set of elements which commute with it. Brauer and Fowler proved:

Theorem

The order of a finite simple group of even order is bounded by a function of the order of the centraliser of an involution; so there are only finitely many simple groups having a given involution centraliser.

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Subsequently, characterising simple groups with a given involution centraliser was a key tool in the proof of CFSG.

The centraliser of an element of a group is the set of its neighbours in the commuting graph. The argument of Brauer and Fowler went, in brief, like this. They give an absolute bound for the diameter of this graph. Then using the fact that involutions have finite valency, they can convert this into a bound for the number of vertices, using graph-theoretic arguments.

Other groups

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On the basis of this and other evidence, Iranmanesh and Jafarzadeh conjectured that there is an absolute bound for the diameter of the commuting graph of any finite group (with the centre removed). Their conjecture was proved for groups with trivial centre by Morgan and Parker; but Giudici and Parker showed that it is false for general groups, where the diameter can be arbitrarily large.

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It may be worth looking at the Giudici–Parker graphs to see if they have interesting graph-theoretic properties.

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The proof goes like this. The conjugacy classes are orbits of the group acting on itself by conjugation; the stabiliser of a point is its centraliser. So the Orbit-Stabiliser Theorem shows that $|C_G(x)| = |G|/|x^G|$, where $C_G(x)$ is the centraliser of x and x^G the conjugacy class containing x.

Since the conjugacy classes partition G, we have

$$|G| = \sum_{i=1}^{k} |G|/|C_G(x_i)|,$$

where x_i runs over a set of conjugacy class representatives. Putting $n_i = |C_G(x_i)|$ and dividing by G, we have

$$\sum_{i=1}^{k} 1/n_i = 1.$$

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Now $|G| = |C_G(1)|$ is the largest value of n_i to occur in a solution arising from a group G. So there are only finitely many possible groups.

Estimates

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The first into the fray were Erdős and Rényi, who showed that $f(n) \geq \log\log n$ (logarithms to base 2). This was improved by Laci Pyber to $\epsilon \log n/(\log\log n)^8$ by Pyber; the exponent 8 was reduced to 7 by Keller, and to $3+\epsilon$ by Baumeister, Maróti and Tong-Viet. It is conjectured that a bound of the form $f(n) \geq C\log n$ holds for some constant C. In the other direction, $f(n) \leq (\log n)^3$.

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I am going to show a different development of Landau's theorem.

The solvable conjugacy class graph

The solvable conjugacy class graph of the group G is defined as follows: the vertices are the conjugacy classes; two classes x^G and y^G are joined if there exist $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is a solvable group.

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The proof uses the Classification of Finite Simple Groups, but in a “light-touch” way. Two open problems are finding a proof not using the Classification, and finding decent bounds for the function involved.

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The power graph and enhanced power graph seem quite similar; indeed, each determines the other. So we expect the difference graph to be relatively sparse. But the analysis I give here could be repeated for many other graphs defined on G, and my guess is that many similar results can be obtained.

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Such groups are called EPPO groups (acronym for "Elements of Prime Power Order"). Their classification has an interesting history; the problem was introduced by Higman and studied by Suzuki. The classification was achieved by Brandl in 1981 and published in a rather obscure journal, so has been rediscovered a number of times.

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Even if these graphs are not equal, they are not too far apart.

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- *The matching numbers of the power graph and enhanced power graph are equal.*

The proof of the last fact, by Swathi, Sunitha and me, follows familiar graph-theoretic arguments about matchings. If we have a matching in the enhanced power graph containing edges not in the power graph, we can replace it by a matching of the same size with fewer edges not in the power graph.

Other group classes defined by graphs

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Other classes defined in similar ways include Dedekind groups (those in which every subgroup is normal), minimal non-abelian groups (or non-nilpotent, or non-solvable), and 2-Engel groups (those satisfying the identity $[[x, y], y] = 1$, where $[x, y]$ is the commutator $x^{-1}y^{-1}xy$.

Finding beautiful graphs

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We computed the generating graph of the alternating group A_5, and asked the computer to tell us the order of its automorphism group. The answer was a shock to us: it was 23482733690880.

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Why are we getting these huge numbers?

Twins

In brief, the explanation is that, if x is an element of order $m > 2$, then for every positive integer d with $\gcd(m, d) = 1$, each of x and x^d is a power of the other; so these two elements have the same neighbours (apart from possibly one another) in most of the interesting graphs on G: power graph, commuting graph, generating graph, $\ldots$

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Call two vertices v, w of a graph Γ twins if they have the same neighbours except possibly for one another. (One can distinguish between open twins, with the same open neighbourhoods, and closed twins, with the same closed neighbourhoods, but we will not require this.)

Now the relation of being equal or twins is an equivalence relation on the vertex set, and the transposition swapping twins and fixing everything else is an automorphism.

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For example, A_5 has ten subgroups of order 3 (so ten twin classes of size 2) and six subgroups of order 5 (so six twin classes of size 4); so we get a group of order $(2!)^{10}.(4!)^6$ fixing the twin classes. These automorphisms are of no interest! How do we strip them away?

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Problem

Choose a type of graph on groups. For which groups G is this graph defined on G a cograph?

A speculation

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As a result of computations, I tentatively identify four types of finite simple groups:

Type 1: These are EPPO groups, whose difference graph has no edges at all. As we have seen, all EPPO groups have been determined; the only simple ones are $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$ for a few small values of q, together with $\mathrm{PSL}(3, 4)$.

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Type 2: Groups for which the difference graph is a cograph, so that twin reduction gives a graph with a single vertex. The simple groups with this property have been determined, up to some number-theoretic problems which will probably not be solved soon. For example, $D(\mathrm{PSL}(2,2^a))$ is a cograph if and only if each of $q-1$ and $q+1$ is either a prime power or the product of two distinct primes.

Type 3: The cokernel of $D(G)$ is a disjoint union of many copies of a small graph. This seems to happen for the remaining groups $\mathrm{PSL}(2, q)$ or $\mathrm{Sz}(q)$. For example, for $q = 23$ or $q = 25$, it consists of 253 or 325 copies of the graph $K_5 - P_4$.

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Type 4: The cokernel of $D(G)$ is an interesting graph. I mention a few examples.

For $G = \mathrm{PSL}(3,3)$, the cokernel of $D(G)$ has vertex set the set of point-line pairs in the projective plane of order 3. There are two types of pair, flags and antiflags: this gives a bipartition of the graph. The antiflag (P,L) is joined to the flag (Q,M) if and only if $Q \in L$ and $O \in M$. This graph has 169 vertices, diameter 5 and girth 6. The valencies of antiflags are 4, those of flags are 9. This simple graph construction may be worth investigating for other projective planes.

For $G = M_{11}$ (the Mathieu group), the cokernel of $D(G)$ is also bipartite, with bipartite sets of sizes 165 and 220; the graph is semiregular, with valencies 4 and 3 respectively in the two partite sets, and has diameter and girth 10.

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In all Type 4 cases I have looked at, going to the cokernel strips away all the unwanted automorphisms: the automorphism group of the graph turns out to be the same as the automorphism group of the group G we start with.

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Further computation would surely find more interesting examples!

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... for your attention.