Graphs on groups and their relations

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International Conference on Algebra and Discrete Mathematics
Savitribai Phule Pune University (SPPU)
27 May 2022

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There are several reasons why we might be interested in graphs like this.

1. The graph gives information about the group. The commuting graph was introduced by Brauer and Fowler in their seminal paper in 1955. They showed that, in a non-abelian finite simple group of even order, elements are not too distant in the graph, and used this to show that there are only finitely many such groups with a prescribed involution centralizer. This was perhaps the first step in the thousand-mile journey to the Classification of Finite Simple Groups.

2. The group gives us constructions of interesting graphs. For example, the power graph (two vertices joined if one is a power of the other) of the Mathieu group M_{11} contains within it some interesting graphs with large girth. This is probably true for other simple groups as well; exploration of this is underway.

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There is far too much known on this topic for a complete account here. I will try to tell the story mostly by examples.

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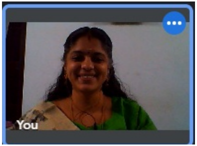

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(In fact the commuting graph is contained in the non-generating graph provided G is either non-abelian or not 2-generated.)

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Other graphs could be added to the list, including the nilpotency graph (g, h joined if they generate a nilpotent group) and the solvability graph (g, h joined if they generate a solvable group).

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- The enhanced power graph is equal to the commuting graph if and only if all Sylow subgroups are cyclic or generalized quaternion. The classification follows from Burnside's transfer theorem and the Gorenstein–Walter theorem.
- For G non-abelian, the commuting graph is equal to the non-generating graph if and only if G is a minimal non-abelian group. These groups were determined by Miller and Moreno in 1904.

Into the second dimension

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This involves choosing also a G-invariant equivalence relation. I will consider the relations of conjugacy and same order. Now given any graph type, say the power graph, we define the conjugacy superpower graph by the rule that g and h are joined if there exist g' and h' in the conjugacy classes of g and h respectively such that g' and h' are joined in the power graph. Similarly for other equivalence relations and other graph types.

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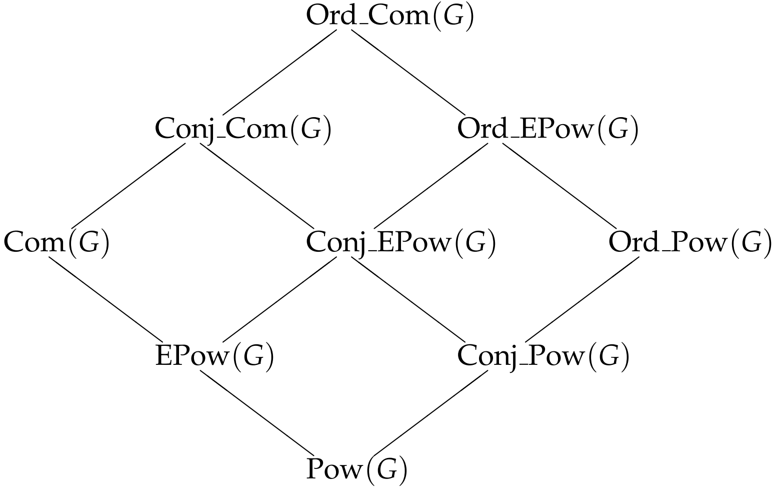

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- *The conjugacy supercommuting graph of G is equal to the commuting graph if and only if G is a 2-Engel group, that is, satisfies the identity* $[x, y, y] = 1$*;*
- *the conjugacy superpower graph of G is equal to the power graph if and only if G is a Dedekind group, that is, one in which every subgroup is normal.*

Comments

Dedekind groups are all known. Such a group is either abelian or of the form $A \times B \times C$ where A is a quaternion group, B an elementary abelian 2-group, and C an abelian group of odd order.

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The only proof we found in the literature was a `StackExchange` post by Korhonen, using a result of Kappe. Information on earlier proofs welcome!

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There are many similar questions which could be asked!

Compressed supergraphs

We defined the supergraphs above to have vertex set the entire group, so that we could compare them with other graphs in the hierarchy. However, it would be more natural to define compressed versions of these graphs, in which we take the vertices to be the equivalence classes of the appropriate equivalence relation (conjugacy or same order, in my examples). Then two classes C and D are joined if there is are vertices $x \in C$ and $y \in D$ which are joined in the original graph.

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In a finite group G, I will denote the conjugacy class of the element g by $g^G = \{x^{-1}gx : x \in G\}$. I will consider the compressed conjugacy superpower graph for the relation of solvability. Thus, two classes g^G and h^G are joined in this graph if there exist $g' \in g^G$ and $h' \in h^G$ such that $\langle g', h' \rangle$ is a solvable group. This is known as the solvable conjugacy class graph, written $\Gamma_{scc}(G)$.

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I shall give the proof. By the Orbit-Stabilizer Theorem, the size of a conjugacy class is given by $|g^G| = |G|/|C_G(g)|$, where $C_G(g)$ is the centralizer of G. Since G is the disjoint union of conjugacy classes, we have

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Now it is an exercise to show that the above equation has only finitely many solutions; and the largest n_i is $|C_G(1)| = |G|$.

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The proof uses the Classification of Finite Simple Groups, but does not require very detailed knowledge about the groups. An important ingredient is a theorem of S. Dolfi, R. M. Guralnick, M. Herzog and C. E. Praeger, extending John Thompson's N-group theorem: A finite group is solvable if and only if its solvable conjugacy class graph is complete.

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Graphs with many pairs of twin vertices are not likely to be so interesting from some points of view. Since my aim in the rest of the talk is to produce beautiful graphs from groups, we will have to deal with the twins.

Twins and twin reduction

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I will call the resulting graph the cokernel of Γ.

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This raises the question: For which finite groups G is the power graph (or one of our other graphs) a cograph? Pallabi Manna will talk about this at the conference.

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- After the above process, we are left with a large number of isomorphic connected components. This occurs for A_7, $\mathrm{PSL}(2, 23)$ and $\mathrm{PSL}(2, 25)$. For example, A_7 gives a graph with 35 connected components, each isomorphic to a tree with three arms of length 3 radiating from a central vertex.

- The graph that remains is connected, often with large girth, and nice structural properties, and with automorphism group equal to the automorphism group of the original simple group. This occurs for $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$ and M_{11}.

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For these three simple groups, the number of vertices, diameter and girth are given in the table.

Group	Vertices	Orbits	Diameter	Girth
$\mathrm{PSL}(3,3)$	754	4	11	12
$\mathrm{PSU}(3,3)$	784	7	10	3
M_{11}	1210	4	20	20

- The graph that remains is connected, often with large girth, and nice structural properties, and with automorphism group equal to the automorphism group of the original simple group. This occurs for $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$ and M_{11}.

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What is going on?

Variants of the generating graph

Recall the generating graph, where two elements are joined if they generate the group. Of course, if the group is not generated by two elements, this graph is null! Fortunately, all finite simple groups are 2-generated, but what about other groups?

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Recently Andrea Lucchini and Daniele Nemmi made two definitions to get around this problem:

- The independence graph of G joins g to h if $\{g, h\}$ is contained in a generating set for G which is minimal (with respect to inclusion).
- The rank graph joins g to h if $\{g, h\}$ is contained in a generating set of minimum cardinality.

Relation to the hierarchy

It is easy to see that the independence graph is contained in the complement of the power graph, while the rank graph is contained in the complement of the enhanced power graph.

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It is easy to see that the independence graph is contained in the complement of the power graph, while the rank graph is contained in the complement of the enhanced power graph. In a preprint which has not yet appeared as far as I know, these two authors, with Colva Roney-Dougal and Saul Freedman, have determined the groups for which equality holds in either of these inclusions.

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… for your attention.