Graphs defined on groups

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What kind of graphs?

To begin: The most important class of graphs on groups is the class of Cayley graphs, but I am not talking about these. My topic is graphs whose definition directly reflects some aspect of the group structure. For example, the commuting graph of a group, where we join two elements x, y of the group by an edge if $xy = yx$.

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This subject dates from 1955, but there has been a big upsurge in activity recently. A couple of years ago, I wrote a survey article with many questions, and put it on the arXiv; Ambat Vijayakumar (Kochi) saw it and asked me to lead an on-line research discussion on the topic. Much of what I have to say arose as a result of this research discussion.

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Similarly, there is work about graphs on other algebraic structures; but my time is short, so I will stick to groups.

An example

Here is a brief example, which I will not pursue. The generating graph of a group has vertex set the non-identity group elements, two elements x and y joined if $\langle x, y \rangle = G$. Now not every group can be generated by two elements; but the Classification of Finite Simple Groups has the consequence that every finite simple group is 2-generated.

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Generating graphs of finite simple groups were shown by Breuer, Guralnick and Kantor to have spread 1; recently, Burness, Guralnick and Harper showed that they have spread 2 (and indeed showed that these two properties are equivalent for generating graphs, and characterised groups having them).

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- Using graphs we can characterise some important classes of groups.
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I hope to show you, at least briefly, examples of all three of these topics.

New results about groups

The classic example of this is the 1955 paper by Brauer and Fowler in which they showed that there are only finitely many finite simple groups of even order which have a given involution centralizer. With hindsight, this was the first step in the thousand-mile journey to the Classification of the Finite Simple Groups. Their proof involved bounding the diameter of the commuting graph of such a group.

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Subsequent authors have bounded the order of such groups; but our extension goes in a different direction.

The solvable conjugacy class graph

Let G be a finite group. The solvable conjugacy class graph of G is the graph whose vertices are the conjugacy classes of non-identity elements of G, two classes C and D adjacent if there exist $x \in C$ and $y \in D$ such that $\langle x, y \rangle$ is a solvable group.

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We do not have a good bound for the order of such a group. Also, our proof uses the Classification of Finite Simple Groups, in a "light-touch" way; we do not know if this can be avoided.

Characterizing classes of groups

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- Choose two different types of graphs defined on groups, and ask for which groups these two graphs coincide.

I will give two examples of the second way. There are results on the first as well: for example, Pallabi Manna, Ranjit Mehatari and I studied groups whose power graph is a cograph (that is, contains no induced 4-vertex path).

Power graph and enhanced power graph

These two graphs have as vertices the elements of G. The power graph of G, two vertices joined if one is a power of the other; in the enhanced power graph, two vertices are joined if both are powers of the same element. Thus, the power graph is a spanning subgraph of the enhanced power graph.

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The Gruenberg–Kegel graph of G has vertices the prime divisors of G, with an edge from p to q if G contains an element of order pq.

A group G is called an EPPO group if every element has prime power order. These were first investigated by Higman in the 1950s, who found the solvable EPPO groups; in the 1960s, Suzuki found the simple ones; and in 1981, Brandl found all these groups.

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In general, these two graphs can be expected to be not very different. For example, Swathi, Sunitha and I showed that they have the same matching number. I will return to this point later.

Super graphs on groups

In a paper with G. Arunkumar, Rajat Kanti Nath and Lavanya Selvaganesh, we proposed the following definition. If Γ is a type of graph defined on groups, then there is a super version of Γ, in which two elements x and y are joined if there exist conjugates x' and y' of x and y which are joined in Γ. (This is the conjugacy supergraph; a similar construction applies for other equivalence relations.)

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A Dedekind group is a group in which every subgroup is normal. Dedekind showed that such a finite group is either abelian, or of the form $Q \times A \times B$, where Q is the quaternion group of order 8, A an elementary abelian 2-group, and B an abelian group of odd order.

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Theorem

Let G be a finite group. Then the power graph and super power graph of G are equal if and only if G is a Dedekind group. The same holds for the enhanced power graph and the super enhanced power graph.

Super graphs, 2

A group G is a 2-Engel group if it satisfies the identity $[x, y, y] = 1$ for all $x, y \in G$, where $[x, y]$ is the commutator $x^{-1}y^{-1}xy$, and $[x, y, z] = [[x, y], z]$.

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A nilpotent group of class 2 satisfies the identity $[x, y, z] = 1$ for all $x, y, z \in G$, so is obviouslyu 2-Engel. In the other direction, Hopkins and Levi independently showed that a 2-Engel group is nilpotent of class 3, and is "close" to being nilpotent of class 2.

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Theorem

The finite group G has commuting graph equal to super commuting graph if and only if G is a 2-Engel group.

This uses two perhaps not well-known equivalents to the 2-Engel property: all centralizers are normal, and conjugate elements commute.

Finding the jewel in the lotus

From some points of view, graphs defined on groups have a lot of irrelevant rubbish; sometimes it is possible to strip it away and reveal some beautiful graphs. The examples I discuss are from a paper with Sucharita Biswas, Angsuman Das and Hiranya Kishore Dey, but I am sure that similar things can be done in many graphs on groups.

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Two vertices x and y of a graph Γ are twins if they have the same neighbours, apart possibly from one another. (Thus there are two kinds of twins; but this will not bother us.) Twin reduction is the process of repeatedly identifying twin vertices until no twins remain. It is not hard to show that the result of this process, up to isomorphism, does not depend on the order of the reductions. I will call this result the cokernel of Γ.

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Proposition

The cokernel of Γ *is the* 1*-vertex graph if and only* Γ *is a cograph.*

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Graphs defined on groups tend to have many twins: if x has order greater than 2, then usually x and x^d are twins for any d coprime to the order of x. So we should apply twin reduction, and reach the cokernel of G.

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We applied this process to the difference graph $D(G)$ of a group G, the graph whose edges are the edges of the enhanced power graph which are not in the power graph. We expect this to be a fairly sparse graph and potentially to contain interesting stuff.

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At this stage, we are doing "experimental mathematics". Empirically, simple groups G seem to fall into four types, as on the next slide.

Finding the jewel in the lotus, 3

- Type 1: G is an EPPO group. Then $D(G)$ has no edges. The simple groups are a few $\mathrm{PSL}(2,q)$ and $\mathrm{Sz}(q)$ together with $\mathrm{PSL}(3,4)$.

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- Type 3: The cokernel of $D(G)$ consists of a large number of isomorphic small graphs, e.g. 253 or 325 copies of $K_5 - P_4$ in $\mathrm{PSL}(2,23)$ and $\mathrm{PSL}(2,25)$ respectively.

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- Type 4: an interesting connected graph typically with large girth. For example, if G is the Mathieu group M_{11}, we obtain a semiregular bipartite graph on $165 + 220$ vertices with valencies 4 and 3, whose diameter and girth are both equal to 10.

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I am sure there is much more of interest to find here. Please have a try!

… for your attention.