A bridge between algebra and combinatorics

Peter J. Cameron, University of St Andrews

Πeter Neumann memorial
9 April 2022

A preprint

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Primitive permutation groups of degree 3p

by Peter M. Neumann.

This paper presents an analysis of primitive permutation groups of degree 3p, where p is a prime number, analogous to H. Wielandt's treatment (19) of groups of degree 2p. It is also intended as an example of the systematic use of combinatorial methods as surveyed in §6 for distilling information about a permutation group from knowledge of the decomposition of its character. The work is organised into three parts. Part I contains the lesser half of the calculation, the determination of the decomposition of the permutation character. Part II contains a survey of the combinatorial methods and, based on these methods, the major part of the calculation. Part III ties up loose ends left earlier in the paper and gives a tabulation of detailed numerical results.

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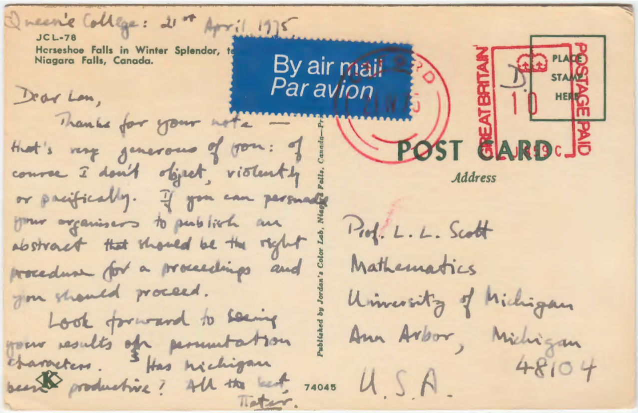
Queen's College: 21st April 1975

JCL-78
Horseshoe Falls in Winter Splendor,
Niagara Falls, Canada.

Dear Len,

Thanks for your note — that's very generous of you: of course I don't object, violently or pacifically. If you can persuade your organisers to publish an abstract that should be the right procedure for a proceedings and you should proceed.

Look forward to seeing your results on permutation characters. Has Michigan been productive? All the best,

Πeter.

By air mail
Par avion

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These types of structure are almost the same, as we will see.

A coherent configuration is a collection $A_1, \ldots, A_r$ of square 0-1 matrices of the same size, summing to the all-1 matrix J and having a subset which sums to the identity matrix I, closed under transposition, and having the property that for any i, j, we have

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Many combinatorial objects are special cases of coherent configurations. The definitions just given probably don't conjure up a picture in your mind. So here is a special case.

Strongly regular graphs

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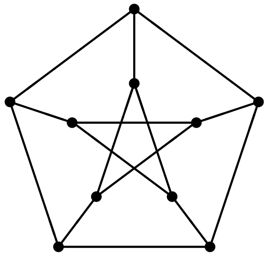
The famous Petersen graph is an example, with $k = 3$, $\lambda = 0$, $\mu = 1$.



Wielandt and Neumann

In 1956, Helmut Wielandt proved that a finite primitive permutation group acting on a set Ω of size $2p$ (where p is an odd prime) is 2-transitive, unless p has the form $2a^2 + 2a + 1$ for some positive integer a, in which case it may have rank 3 (this means three orbits on the set $\Omega \times \Omega$, whose sizes are expressed in terms of the parameter a.)

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Wielandt needed to do a lot of work decomposing the permutation character of his group, and then the combinatorial argument, though innovative, is fairly straightforward. For Neumann, on the other hand, the decomposition of the permutation character was easier, because of a theorem of Walter Feit proved in the meantime; but the combinatorial part is much more complicated, and the result too; there are three possible quadratic expressions for the prime p as well as three sporadic values.

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Whereas Higman and Neumann considered the coherent configuration associated with a permutation group, which takes all orbital graphs together and uses numerical and algebraic information, Sims chose a particular graph and went more deeply into its structure.
This led him to his celebrated conjecture, later proved, using the Classification of Finite Simple Groups (CFSG) by three of Πeter's students together with Gary Seitz.

Permutation groups and combinatorics

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The two subjects are now close partners.

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As I explained, Πeter's paper was never published. However, it is a *tour de force*, and had it been published it would have been recognised as an important link in the chain of ideas sketched earlier. I believe that it is too good to be lost.

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But that is not the end of the story …

As noted, Wielandt first showed that the permutation character decomposes into irreducible constituents of degrees 1, $p - 1$, and p. From general theory, these numbers are the multiplicities of the eigenvalues of the matrices in the corresponding coherent configuration (these are the identity and the adjacency matrices of a strongly regular graph and its complement).

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In the case $p = 5$, the strongly regular graph is the famous Petersen graph, which we met earlier.

In fact the combinatorial part of Wielandt's argument shows the following:

Theorem

Let Γ be a strongly regular graph on $2n$ vertices, whose eigenvalues have multipicities 1, $n-1$ and n, for some natural number n. Then one of the following is true:

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I am not sure who first noticed this. The proof is in my book with Jack van Lint. Note that in the second case, the Petersen graph and its complement are not the only examples; there are a number of further examples (the first pairs having 26 vertices).

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