Graphs defined on groups

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Background

I will begin by introducing some of the main characters in the story.

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Algebraic graph theory is the area where these two very different subjects can meet and have a productive relationship.

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I will not do so; I will explain why shortly. So for me the vertex set of the commuting graph is G.

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- As noted, they don't use the word graph, but they make extensive use of the graph distance, the length of the shortest sequence from x to y not containing the identity, where consecutive elements commute. Questions about connectedness and diameter of this graph now have an extensive literature.
- The main result of the paper is that, given a group H with a central involution, there are only finitely many finite simple groups having an involution whose centraliser is H. This result was fundamental to the Classification of Finite Simple Groups; their paper was perhaps the first step on this thousand-mile journey.

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This is not the complete *dramatis personae*, just the big stars. Some bit players will come in later. Indeed you can imagine some for yourself. Noting that x and y are joined in the commuting graph if and only if $\langle x, y \rangle$ is abelian, we could define a graph where the joining rule is $\langle x, y \rangle$ is nilpotent, or solvable, or ...

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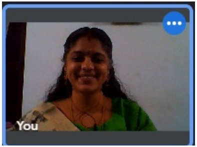

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In order to get started, we observe that these graphs form a hierarchy; each is contained in the next as a spanning subgraph. This is the main reason for taking the vertex set in each case to be the whole group.

A hierarchy of graphs

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This innocent question leads to some deep and important group theory. For example, a paper in preparation by Freedman, Lucchini, Nemmi and Roney-Dougal (which I won't have time to discuss).

The hierarchy

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Each is contained in the next, except that the commuting graph is contained in the non-generating graph if and only if G is either non-abelian or has more than two generators (that is, for all groups except 2-generated abelian groups).

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- The power graph is null if and only if G is the trivial group (for the identity is joined to all other vertices).
- The non-generating graph is complete if and only if G is not 2-generated.
- The commuting graph is equal to the non-generating graph if and only if G is a minimal non-abelian group. Such groups were determined by Miller and Moreno in 1904.

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The groups in each of these classes have been determined. Before explaining this, let me mention another graph associated with a finite group.

The Gruenberg–Kegel graph

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- *G is a Frobenius or 2-Frobenius group; or*
- *G is an extension of a nilpotent π-group by a simple group by a π-group, where π is the set of primes in the connected component containing 2.*

EPPO groups

The group G is an EPPO group ("Elements of Prime Power Order") if every element of G has prime power order. These groups were studied by Higman in the 1950s; he determined the solvable ones. Following the discovery of his infinite family of simple groups, Suzuki was able to determine the simple EPPO groups. Subsequently Brandl gave a complete classification, which was rediscovered by several authors.

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- In the cyclic case, using Burnside's transfer theorem, G is metacyclic (i.e., has a cyclic normal subgroup with cyclic quotient).
- If the Sylow 2-subgroups are generalized quaternion, then using Glauberman's Z^*-theorem and the Gorenstein–Walter theorem, G has a normal subgroup N of odd order; G/N has a unique involution z, and the quotient by $\langle z \rangle$ is a known group.

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There are plenty of open questions here; the only case to have been looked at (as far as I know) is the power graph and enhanced power graph. Again not many results are known.
Recall that these graphs are equal for a group G if and only if every element of G has prime power order.

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One slightly surprising thing about the second result is that we do not have a formula for the matching number of $\mathrm{Pow}(G)$ for an arbitrary group G. The theorem is proved by showing that, given any matching in $\mathrm{EPow}(G)$, we can find another matching of the same size which has fewer edges which don't belong to $\mathrm{Pow}(G)$.

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Otherwise either q or $q + 1$ is prime, giving the remaining cases. So our problem includes the determination of all Fermat and Mersenne primes!

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I will look at one further property to illustrate the benefit of treating the graphs as a hierarchy.

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If Γ is the comparability graph of a finite partial order, then there is a finite group G such that Γ is isomorphic to an induced subgraph of $\mathrm{Pow}(G)$.

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But using our hierarchy, we can strengthen the last result.

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Suppose that the edges of a finite complete graph are coloured blue, yellow and white in any manner. Then the vertex set can be embedded into a finite group G such that

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- ignoring the yellow-white distinction, enhanced power graphs form a universal class;
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- ignoring the blue-yellow distinction, the class of graphs of the form $(\mathrm{Com} - \mathrm{EPow})(G)$ is universal.

A final topic

There is much much more that I haven't talked about, and many many open problems. Please see the references, or email me if you want to discuss some of this or work on some open problems.

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I will finish with a topic from the sheaf of results that have been proved as a result of the research discussion group; this has some cute mathematics …

Clique number of the power graph

As a final topic, there is a sense in which the enhanced power graph is not much larger than the power graph. For example, while $\omega(\mathrm{Pow}(G)) \leq \omega(\mathrm{EPow}(G))$, it is true the $\omega(\mathrm{EPow}(G))$ is bounded above by a function of $\omega(\mathrm{Pow}(G))$. This can be seen by looking more closely at the clique number of $\mathrm{Pow}(G)$.

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Any edge of $\mathrm{Pow}(G)$ is contained in a cyclic subgroup; and if every pair of vertices of a set S in a group are contained in a cyclic subgroup, then S is contained in a cyclic subgroup. So $\omega(G)$ is equal to the maximum of $\omega(C)$ over all cyclic subgroups C of G.

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Similarly, $\omega(\mathrm{EPow}(G))$ is equal to the order of the largest cyclic subgroup of G.

So it suffices to look at cyclic groups.

In a cyclic group

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From this it follows easily that $f(n) \leq 3\phi(n)$. Hence n is bounded above by $cm \log\log m$, where $m = f(n)$; and the same bound holds for the clique numbers m and n of the power graph and enhanced power graph of an arbitrary group.

In fact,

$$\limsup f(n)/\phi(n) = 2.6481017597\ldots,$$

where the constant on the right is

$$\sum_{k\geq 0}\prod_{i=1}^{k}\frac{1}{p_i - 1},$$

where $p_1, p_2, \ldots$ are the primes in order.

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This suggests several questions, such as

- is this constant rational, algebraic or transcendental?
- what other numbers are limit points of the set $\{f(n)/\phi(n) : n \in \mathbb{N}\}$?

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Thank you . . .

. . . for your attention.