Finding the jewel in the lotus

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This is part of my ongoing project on "graphs defined on groups", but strikes out in a rather different direction.

By way of introduction, I mention an experience with the generating graph of a finite group: the vertices are the non-identity group elements, two vertices x and y joined if $\langle x, y\rangle = G$. When Colva and I were looking at this graph, we asked the computer to tell us how many automorphisms the generating graph of the alternating group A_5 had. We were somewhat surprised at the answer: 23482733690880. Why so big?

Explaining this gives us a way to strip away a plethora of irrelevant detail from such a graph and, if we are lucky, find a beautiful and useful graph within. That is what this talk is about.

1. Twin reduction

Two vertices of a graph are *twins* if they have the same neighbours apart from one another. There are two types of twins: open twins (when the open neighbourhoods are equal) and closed twins (when the closed neighbourhoods are equal), though we don't really need this distinction. Being equal or twin is an equivalence relation on the vertex set.

If two vertices are twins, then the permutation which transposes them and fixes everything else is a graph automorphism. Hence the automorphism group has a normal subgroup which is the direct product of symmetric groups on the twin classes.

Twin reduction is the process where we pick a pair of twins and identify them (or, if you prefer, throw one away), and repeat the process until no pairs of twins remain. It is an easy exercise to show that the result of twin reduction is, up to isomorphism, independent of the way in which the reduction is done.

To give it a name, I will call the result of this reduction the *cokernel* of the graph. No connection with cohomology theory is intended!

For example, take a 4-cycle. The two pairs of opposite vertices are open twins; reducing these converts the graph to a single edge, whose vertices are closed twins, so one more step brings us to the graph consisting of a single vertex, which is the cokernel.

A graph is a *cograph* if and only if it does not contain the 4-vertex path as an induced subgraph. Cographs form an important class of graphs, and have arisen in many different places, and been given many different names: "cographs" is the shortest. They also have many different characterisations. For example, they form the smallest class of graphs that can be built from one-vertex graphs by the operations of disjoint union and complementation. The one I will need is the following:

Theorem 1 *A finite graph is a cograph if and only if its cokernel is the* 1*-vertex graph.*

2. The difference graph of a finite group

A large number of different graphs have been defined on a group using the group structure; the oldest of these is the *commuting graph*, in which two elements are joined if they commute. This was used by Brauer and Fowler in a paper fundamental to the finite simple group classification. I will just define what I need here. In each case the vertex set is the group G.

- The *power graph*: $x \sim y$ if one of x and y is a power of the other.
- The *enhanced power graph*: $x \sim y$ if and only both x and y are powers of an element z.
- The edge set of the power graph is contained in that of the enhanced power graph; the *difference graph* $D(G)$ has edges those of the enhanced power graph not contained in the difference graph.

Now it is an exercise to show that in a group of prime power order, the power graph and enhanced power graph are equal, so the difference graph has no edges. What about the converse?

Theorem 2 *The difference graph of G has no edges if and only if every element of G has prime power order.*

Such groups are sometimes called *EPPO groups*. After work by Graham Higman (who classified the soluble ones) and Michio Suzuki (who classified the simple ones), the complete list was published by Rolf Brandl in 1981, though his work was rediscovered by a number of authors.

Here I am primarily concerned with simple groups.

Theorem 3 *The simple EPPO groups are* $\mathrm{PSL}(2,q)$ *for* $q = 4,7,8,9,17$, $\mathrm{Sz}(q)$ *for* $q = 8,32$, *and* $\mathrm{PSL}(3,4)$.

These groups are of course of no use if we are looking for cases where the difference graph is an interesting graph! We also need to discard those graphs where the difference graph is a cograph:

Theorem 4 *The simple groups for which the difference graph is a cograph are*

- $\mathrm{PSL}(2,q)$, *where each of* $(q-1)/\gcd(q-1,2)$ *and* $(q+1)/\gcd(q+1,2)$ *is either a prime power or the product of two distinct primes;*
- $\mathrm{Sz}(q)$, *where* q *is an odd power of* 2 *and each of* $q-1$, $q+\sqrt{2q}+1$ *and* $q-\sqrt{2q}+1$ *is either a prime power or the product of two primes.*

We do not know whether there are finitely or infinitely many groups covered by this theorem.

3. Towards the interesting examples

For the remaining simple groups G, it seems (though I have no proof) that one of two things happens for Γ, the cokernel of $D(G)$;

- Γ has many connected components which fall into one or two isomorphism types, probably not very interesting (the remaining groups $\mathrm{PSL}(2,q)$ and $\mathrm{Sz}(q)$ tend to do this); or
- Γ is connected, its automorphism group is just $\mathrm{Aut}(G)$, and it is a very interesting graph from various points of view.

I will just discuss a couple of examples.

- Let $G = \mathrm{PSL}(3,3)$. Then the cokernel of $D(G)$ is the following graph defined in the projective plane of order 3. The vertices are the ordered pairs (P, L) where P is a point and L a line. These are of two types: *flags* (where P and L are incident) and antiflags (where they are not). All edges join a flag to an antiflag; the antiflag (P, L) is incident with the flags (Q, M) where P is incident with M and Q with L. This graph has diameter 5 and girth 6, and 169 vertices.
- Let G be the Mathieu group M_{11}. Then the cokernel of $D(G)$ is a graph on 385 vertices; it is bipartite, with parts of size 165 and 220 (each of these sets an orbit of the automorphism group, which is just M_{11}), and has diameter 10 and girth 10. The valencies of vertices in the two parts are 4 and 3 respectively.
- The (non-simple) Ree group $R_1(3) \cong \mathrm{P\Gamma L}(2,8)$ gives a semiregular bipartite graph on 63+84 vertices, with valencies 4 and 3, with diameter 5 and girth 6.

Surely there are other interesting examples here. Note that the cokernel is much smaller than the graph we started with, and tends to have small valency and large girth (good network properties).