Permutation groups and transformation semigroups: 1. Permutation groups

Peter J. Cameron, University of St Andrews

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I’ll know my song well before I start singing

Bob Dylan

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I am going to tell you about some aspects of this.

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We say G is transitive if, for all $\alpha, \beta \in \Omega$, there is an element $g \in G$ satisfying $\alpha g = \beta$. If G is transitive, then we say it is primitive if the only G-invariant partitions of Ω are the trivial ones (the partition into singletons, and the partition with a single part).

Higman's Theorem

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Note that we can construct G-invariant graphs by taking orbits of G on pairs of elements of Ω as edges. These are the orbital (di)graphs.

Multiple transitivity

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Clearly t-transitivity implies t-homogeneity. If $5 \leq t \leq n/2$, a beautiful theorem of Livingstone and Wagner asserts that the converse is true. All t-homogeneous but not t-transitive groups for $t = 2, 3, 4$ were found by Kantor (before CFSG).

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The classification of t-transitive groups for $t \geq 2$ had to wait for CFSG (the Classification of Finite Simple Groups before it could be completed; but now we have a complete list of such groups.

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A virtue of this definition is that, for any class $\mathcal{C}$, the class of $\mathcal{C}$-free permutation groups is closed upwards.

Playing the game

We will see several examples. However, the way to play this game is not to think up an arbitrary class $\mathcal{C}$ and examine the $\mathcal{C}$-free or $\mathcal{C}$-closed permutation groups. Rather, we have a property of permutation groups we want to study; understanding the $\mathcal{C}$-free or $\mathcal{C}$-closed structures for an appropriate class is likely to help the investigation. Even better are cases when we can build arbitrary permutation groups from the $\mathcal{C}$-free groups.

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Note that if G is not $\mathcal{C}$-free then it preserves a non-trivial $\mathcal{C}$-structure. The nicest cases are those where we can use this to get a reduction for G, and understand it in terms of smaller permutation groups. This is the case for transitivity and primitivity, for example.

How it works

Let $\mathcal{C}$ be the class of "subsets": a $\mathcal{C}$-object is a subset of Ω. The only subsets invariant under the symmetric group are the empty set and Ω; so G is $\mathcal{C}$-free if and only if it is transitive.

How it works

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Again, let $\mathcal{C}$ be the class of “directed graphs”. A directed graph invariant under the symmetric group is either null or complete; so, if G is $\mathcal{C}$-free, then any pair of distinct points can be mapped to any other pair by an element of G (otherwise an orbit of G would be a digraph which is neither complete nor null); in other words, G is 2-transitive.

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Digraphs	2-transitive
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Another class $\mathcal{C}$ we have just begun to study consists of poset block structures, where the $\mathcal{C}$-closed groups are the generalised wreath products.

Two challenges

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For semigroup theorists:

Question

Can we define interesting classes of (partial) transformation monoids in this way?

Reductions

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If G is transitive but imprimitive, it preserves a partition, and is embedded in the wreath product $H \wr K$, where H is the group induced on a block of the partition by its setwise stabiliser, and K the group induced on the set of parts of the partition. This is the imprimitive action of the wreath product.

Hamming graphs and basic groups

Let m, q be integers greater than 1. The Hamming graph $H(m, q)$ is the graph whose vertices are all words of length m over an alphabet of size q (so it has q^m vertices. A primitive group which preserves a Hamming graph is contained in the wreath product of the group (of degree q) induced on the symbols occurring in a given position by the stabiliser of that position in G and the group of permutations on the set of coordinate positions induced by G (of degree m).

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A primitive group is basic if it preserves no Hamming graph with $m, q > 1$. Thus, a group which is primitive but not basic is embeddable in a wreath product (in its product action).

Two special types of group

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A subgroup of $\mathrm{AGL}(V)$ containing T is the semidirect product of T with a subgroup H of $\mathrm{GL}(V)$. It is necessarily transitive, since T is; it is primitive if and only if H is an irreducible linear group; and it is basic if and only if H is a primitive linear group, one which preserves no non-trivial direct sum decomposition of V.

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I will not give the rather involved definition of a diagonal group here; suffice to say that the diagonal group $D(H, m)$ depends on a group H and a positive integer m; it has degree $|H|^m$ and has a normal subgroup H^{m+1} acting on the cosets of a diagonal subgroup, the quotient contained in the group generated by $\mathrm{Aut}(H)$ and the symmetric group S_{m+1}.

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Since affine groups preserve affine spaces, and diagonal groups preserve structures called diagonal semilattices, we can say that a permutation group which preserves no non-trivial subset, partition, Hamming graph, affine space, or diagonal semilattice is almost simple.

Finite simple groups

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- *one of the 26 sporadic groups.*

It follows from CFSG that, if S is one of these groups, then $\mathrm{Aut}(S)/S$ is very small (and in any case soluble). The combined efforts of many mathematicians has led to a good understanding of simple (and almost simple) groups, such as knowledge of their maximal subgroups and linear representations.

Applications

The classification of 2-transitive groups follows from this. A 2-transitive group is clearly primitive and basic, and it is not hard to show that diagonal groups cannot be 2-transitive. So these groups are affine or almost simple; and using knowledge of the almost simple groups and their representations, a complete list can be found. (In fact, much less than the full strength of O'Nan–Scott is needed here; the reduction is due to Burnside.)

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More generally, Wielandt intruduced the class of $\frac{3}{2}$-transitive groups, those which are transitive and the stabiliser of a point α has all remaining orbits of the same size. (This class is not upward-closed so cannot be included in our general scheme.) Wielandt showed that a $\frac{3}{2}$-transitive group is either primitive or a Frobenius group, a group in which all 2-point stabilisers are trivial. Any Frobenius group is $\frac{3}{2}$-transitive; the primitive ones have been classified, using CFSG.

Low in the hierarchy

The properties we have examined so far are almost all at least as strong as primitivity. I want to conclude with several properties which are weaker, which I have investigated with Marina Anagnostopoulou-Merkouri and, in part, with Enoch Suleiman and Rosemary Bailey.

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Peter Neumann pointed out that in the Second Memoir, Galois sometimes confused the notions of primitivity and quasiprimitivity.

Pre-primitivity

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We have various results about such groups, including the fact that a wreath product of transitive groups is pre-primitive if and only if the factors are.

Invariant partitions

The set of partitions of Ω forms a lattice: the meet of two partitions P and Q is the partition whose parts are all non-empty intersections of parts of P and Q, and the join is the partition into connected components of the graph in which two points are adjacent if and only if they are in the same part of either P or Q.

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In addition, a partition is uniform if all parts have the same size, and two partitions commute if the corresponding equivalence relations do.

Statisticians define an orthogonal block structure to be a sublattice of the partition lattice consisting of pairwise commuting uniform partitions (containing the two trivial partitions). Any OBS is a modular lattice; a poset block structure is a distributive OBS.

OB and PB permutation groups

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Generalized wreath product

PB groups are related to another concept, which I cannot describe in detail. It is well-known that a finite distributive lattice is the lattice of down-sets in a finite poset. There is a concept of generalized wreath product defined by a poset with a permutation group at each element.

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For example, there are two 2-element posets. Suppose that groups H and K are given at the two points. If the poset is an antichain, the GWP is the direct product; if it is a chain, with H above K, the GWP is the wreath product $K \wr H$.

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Indeed, we expect to be able to replace the symmetric group by appropriate subgroups induced by the action of G; but this is work in progress.

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