Regular polytopes of high rank for symmetric groups

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The authors

This is joint work with Maria Elisa Fernandes (Aveiro) and Dimitri Leemans (Brussels).

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The abstract structure of the polytope is given by the incidence and the order. Reversing the order, retaining the incidence, gives the dual polytope.

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There is also a connectedness condition, which I will not define precisely (but we will see its effect).

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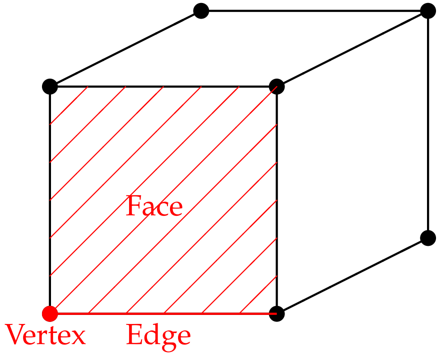

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If the polytope is regular, then there is a unique automorphism s_i which maps F to F_i; ts square fixes F so is the identity. Thus, $s_0, s_1, \ldots, s_{r-1}$ are involutions; s_0 interchanges the two vertices on the edge in F; s_1 interchanges the two edges incident with the vertex and face of F; and so on.

Generation

It follows from the connectedness condition that the automorphism group of the polytope is generated by the r involutions $s_0, s_1, \ldots, s_{r-1}$. So the automorphism group of the polytope is a group generated by r involutions, hence a quotient of a Coxeter group.

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Let us see how this works in our example.

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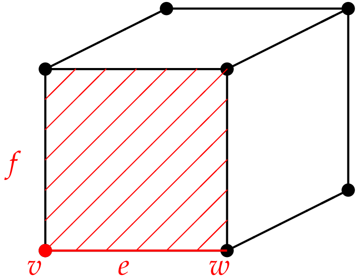

Here s_0 should interchange v and w; s_1 should interchange e and f; and s_2 should interchange the front face with the bottom face.

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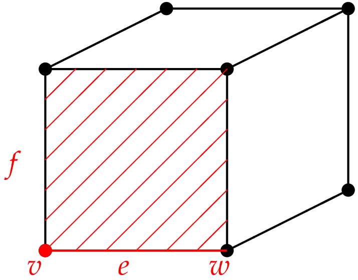
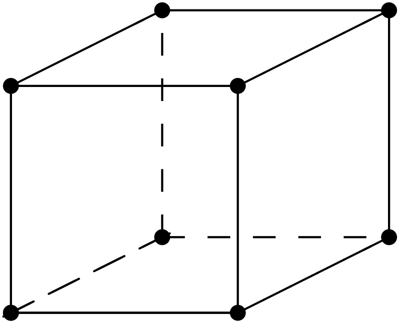
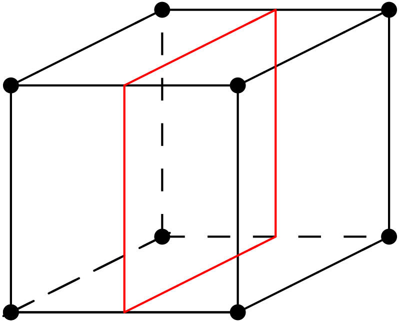
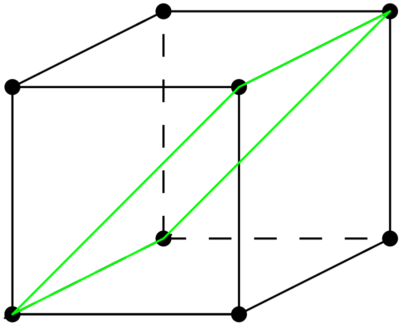


Here s_0 should interchange v and w; s_1 should interchange e and f; and s_2 should interchange the front face with the bottom face. In a general polytope there is no reason for such a global symmetry to exist; but the cube is a regular polytope . . .

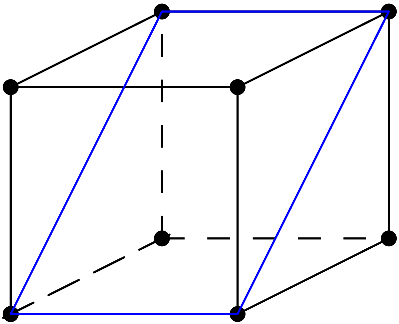




The map s_0 is reflection in the red mirror.



The map s_0 is reflection in the red mirror.
The map s_1 is reflection in the green mirror.



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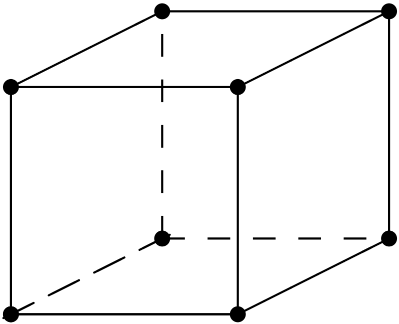

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These reflections satisfy the Coxeter relations for the group $C_2 \times S_4$:

$$\langle s_0, s_1, s_2 \mid s_0^2 = s_1^2 = s_2^2 = (s_0 s_1)^4 = (s_0 s_2)^2 = (s_1 s_2)^3 = 1 \rangle.$$

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A group G generated by involutions $s_0, \ldots, s_{r-1}$ satisfying these properties is called a string C-group. Thus the automorphism group of a regular polytope is a string C-group; and conversely, from a string C-group a construction of Jacques Tits produces a regular polytope, unique up to isomorphism and duality (reversing the partial order, or reversing the order of the generating involutions).

String C-groups for S_n

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It is easy to see that the elements $s_0, s_1, \ldots, s_{r-1}$ are independent, in the sense that none of them is in the group generated by the others.
A theorem of Julius Whiston asserts that an independent set in S_n has cardinality at most $n-1$, with equality only if it generates the group. Philippe Cara and I found all the independent generating sets of size $n-1$ for S_n.

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The corresponding polytope is the regular simplex.
Moreover, this is the unique polytope of rank $n-1$ with group S_n, up to isomorphism, for $n \geq 5$; this is easily read off from my results with Cara, since the only case with $n \geq 5$ in which the generators are all involutions is when they are the edges of a tree, and the string condition forces this tree to be a path.

Ranks $n-2$, $n-3$, $n-4$

Building on this, Fernandes and Leemans showed that there is a unique string C-group for S_n of rank $n-2$ for $n \geq 7$ (up to isomorphism and duality). The corresponding polytope is a generalized petrial of the hypercube (a skew polytope built from the petrie polytope of the cube's vertex figure).

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The obvious conjecture is what we have just proved.

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If c_k denotes this number, then the first six values of c_k are

$$1, 1, 7, 9, 35, 48, 135.$$

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We do not know the next term. It would suffice to count the string C-groups of rank 11 for S_{19}, but S_{19} is quite a big group!

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The last of the three results given was proved in Aveiro, where the photo I showed earlier was taken.

Fracture graphs

Thus we may assume that, if we have a large rank (greater than $n/2 + c$) string C-group representation for S_n, with generators $s_0, \ldots, s_{r-1}$, then the maximal parabolic subgroups

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Use of this graph, which was pioneered in some of the earlier work, is a crucial tool in the argument.

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Fracture and 2-fracture graphs are not unique; but this gives the freedom to modify such a graph into one more suitable for our purpose.
In the regime where we are most interested, the rank is about $n/2$, and the number of edges in a 2-fracture graph is twice the rank, so these graphs are close to trees (often all components are either trees or unicyclic). If there are cycles, we can move them around by replacing one edge with another.

Splits and perfect splits

For the next part it might help you to think about the Moore generators of S_n:

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We say that index i is a split for a string C-group $C \leq S_n$ if the domain $\{1,\ldots,n\}$ can be partitioned into two parts O_1 and O_2 such that s_i is the unique involution interchanging points in different parts, and there is at most one such pair of points interchanged.

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If i is a split, then we can write $s_j = t_j u_j$ for $j \neq i$, where t_j acts on O_1 and u_j on O_2; and $s_i = t_i(\alpha,\beta)u_i$, where $\alpha \in O_1$ and $\beta \in O_2$. If $t_j = 1$ for $j > i$ and $u_j = 1$ for $j < i$ (in other words, if $s_0,\ldots,s_{i-1}$ act only on O_1 and $s_{j+1},\ldots,s_{r-1}$ only on O_2), we call the split perfect.

Rank and degree extensions

Now suppose that i is a perfect split for a string C-group on S_n. We construct a string C-group on S_{n+1} as follows. Take a new element γ in the domain. Now replace the generator $s_i = t_i(\alpha, \beta)u_i$ by two generators

$$s_i' = t_i(\alpha, \gamma), \quad s_i'' = (\gamma, \beta)u_i.$$

We have increased both the degree and the rank by 1, so that the difference remains the same.

Proof of the theorem

Now it can be shown that this extension gives a bijection from string C-groups of rank $n-k$ with group S_n and a perfect split to string C-groups of rank $n-k+1$ with group S_{n+1} with a perfect split.

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Now it can be shown that this extension gives a bijection from string C-groups of rank $n-k$ with group S_n and a perfect split to string C-groups of rank $n-k+1$ with group S_{n+1} with a perfect split.
The difficult part of the proof involves showing that, if $n \geq 2k+3$, then a string C-group of rank $n-k$ with group S_n has a perfect split. This requires many pages of detailed argument with fracture and 2-fracture graphs.

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This proves the theorem, and shows that indeed to compute the kth term in the sequence we only have to classify the string C-groups for S_{2k+3} of rank $n-k = k+3$.

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S_n	Rk $n-1$	Rk $n-2$	Rk $n-3$	Rk $n-4$	Rk $n-5$	Rk $n-6$
S_5	1	4				
S_6	1	4	2			
S_7	1	1	7	35		
S_8	1	1	11	36	68	
S_9	1	1	7	7	37	129
S_{10}	1	1	7	13	52	203
S_{11}	1	1	7	9	25	43
S_{12}	1	1	7	9	40	75
S_{13}	1	1	7	9	35	41
S_{14}	1	1	7	9	35	54
S_{15}	1	1	7	9	35	48
S_{16}	1	1	7	9	35	48

Question

What about alternating groups? The maximum rank of a polytope for A_n is known to be $\lfloor (n-1)/2 \rfloor$ for $n \geq 12$, but we have no characterisation of string C-groups achieving or close to this bound.

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Question

What about other groups?

The maximal size of a minimal generating set (the analogue of Whiston's result) gives an upper bound, but things are much more difficult in general. I have a related conjecture whose proof depends on a question about subgroup lattices:

Question

What about alternating groups? The maximum rank of a polytope for A_n is known to be $\lfloor (n-1)/2 \rfloor$ for $n \geq 12$, but we have no characterisation of string C-groups achieving or close to this bound.

Our construction increasing both rank and degree by 1 will be of no use; we need to increase degree by 2 and rank by 1.

Question

What about other groups?

The maximal size of a minimal generating set (the analogue of Whiston's result) gives an upper bound, but things are much more difficult in general. I have a related conjecture whose proof depends on a question about subgroup lattices:

Question

Is it true that the maximum size of an independent set in G is equal to the maximum, over all permutation representations, of the maximum size of a minimal (under inclusion) base for G?

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Question

Do the polytopes have nice geometric realisations?

If you are interested, our paper is on the arXiv, 2212.12723.

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… for your attention.