Sylvester designs

Peter J. Cameron, University of St Andrews

Midsummer Combinatorics Workshop 2023

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I liked it so much that it is a bit surprising that I went on to group theory and combinatorics rather than topology and analysis.

Two kinds of mathematics

In the preface, he says,

> *It seems to me that a worthwhile distinction can be drawn between two types of pure mathematics. The first—which unfortunately is somewhat out of style at present—centers attention on particular functions and theorems which are rich in meaning and history, like the gamma function, or on juicy individual facts, like Euler's wonderful formula*
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Most of us here, I think, are more comfortable with the second kind of mathematics. But today I will give you an example of the first.

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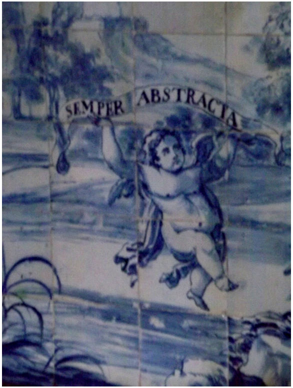

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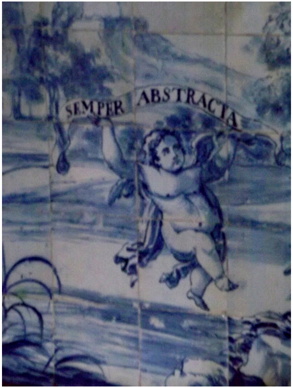

But here I will not follow that advice …

My topic

The topic is taken from the paper

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I will start with a little of the background.

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In this case, the way we allocate varieties to plots has an effect on the amount of information we can extract.

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The designs which achieve this are called A-, D-, E-optimal respectively.

The concurrence graph

The associated parameters are determined by the Laplacian spectrum of a certain multigraph called the concurrence graph. The vertices of the graph are the varieties or treatments being tested; the number of edges between vertices v_i and v_j is the number of occurrences of v_i and v_j in the same block. (For example, a block containing three occurrences of v_i and two of v_j contributes 6 to this number.)

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The Laplacian matrix has (i, j) entry the negative of the number of edges from v_i to v_j if $i \neq j$, and the valency of v_i if $i = j$. It is positive semidefinite; the multiplicity of 0 is the number of connected components.

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I make one further assumption: the design should be equireplicate, that is, each treatment should be used the same number (say r) of times.

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The *D*-optimal design maximizes the number of spanning trees, and hence the geometric mean of the nontrivial Laplacian eigenvalues.
The *E*-optimal design maximizes the smallest non-trivial eigenvalue. This important parameter is connected with isoperimetric number and rate of convergence of a random walk.

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All these designs are optimal on all criteria, and are widely used in practice. They also have the good feature that, if one replicate is lost for some reason, the remaining $r - 1$ replicates still carry an (optimal) square lattice design (provided that $r > 2$).

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I will briefly describe this “juicy individual fact”, using Sylvester’s quaint terminology.

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The symmetric group acts differently on A and X, giving rise to an outer automorphism.

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So there is a natural transformation from Φ^2 to the identity.

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Related to the last is the construction of a graph which Norman Biggs called the Sylvester graph. Let X be the set of totals on A. The vertex set of the graph is $A \times X$; there is an edge from (a, x) to (b, y) if and only if the duad $\{a, b\}$ belongs to the syntheme $x \cap y$.

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This is a distance-transitive graph of valency 5 on 36 vertices; its automorphism group is equal to the automorphism group of S_6, with order 1440. Its diameter is 3, and two vertices are at distance 3 if and only if they agree in one coordinate (same point of A or same total).

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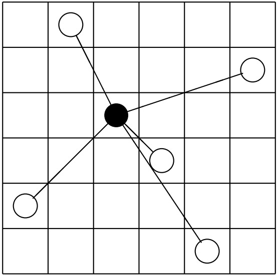

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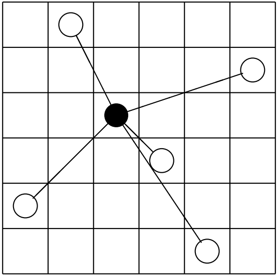

A galaxy of starfish consists of all the starfish whose "heads" lie in a particular column. Our comment about distance 3 shows that the starfish in a galaxy are pairwise disjoint and so partition the set of vertices.

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Two points on an edge of the graph are contained in two blocks, any other pair of points is in just one block. So the concurrence matrix is the sum of the adjacency matrix of the Sylvester graph and the all-1 matrix (with a multiple of I subtracted), and its eigenvalues are easily calculated.

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All these designs do very well on the optimality criteria (especially A, the most commonly used in this context).

Two more

Emlyn Williams maintains a program `CycDesign`, which uses methods including random search to find good block designs. After we announced the Sylvester design, he used his software to find another design with the same value on the A-criterion to four places of decimals. It turns out that it has exactly the same concurrence matrix, and so agrees on all criteria; but it is not the same design, since its automorphism group is trivial.

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Two problems

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The second problem may be difficult. We do not know whether the three examples we know are the only ones, or whether there are billions of designs, or anywhere in between.