What can graphs and algebraic structures say to each other?

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Graphs and algebraic structures

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Much of this concerns groups and rings, but there are also connections with semigroups, vector spaces, etc. Just on groups, the graphs that have been studied include power graph, commuting graph, generating graph and several variants; also "super" and "contracted" versions of these. For rings we have the zero-divisor graph, unit graph, and others.

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Can we extract any general principles?

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In what follows, I will talk mostly about groups.

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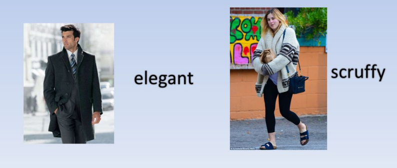

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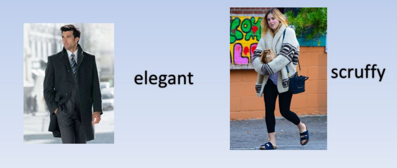

We would expect to find that graphs associated with algebraic structures are less scruffy than general graphs. Later I will show you some examples of beautiful graphs from groups.

An example

Here is a brief example, which I will not pursue. The generating graph of a group has vertex set the non-identity group elements, two elements x and y joined if $\langle x, y \rangle = G$. Now not every group can be generated by two elements; but the Classification of Finite Simple Groups has the consequence that every finite simple group is 2-generated.

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Generating graphs of finite simple groups were shown by Breuer, Guralnick and Kantor to have spread 1; recently, Burness, Guralnick and Harper showed that they have spread 2 (and indeed showed that these two properties are equivalent for generating graphs, and characterised groups having them).

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There has been a lot of work on computing many different graph-theoretic parameters of some of the graphs. This is important work, but I regard it as more in the nature of filling in detail in the background of the picture.

New results about groups

The classic example of this is the 1955 paper by Brauer and Fowler in which they showed that there are only finitely many finite simple groups of even order which have a given involution centralizer. With hindsight, this was the first step in the thousand-mile journey to the Classification of the Finite Simple Groups. Their proof involved bounding the diameter of the commuting graph of such a group.

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Theorem

Given a positive integer k, there are only finitely many finite groups which have exactly k conjugacy classes.

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Let G be a finite group. The solvable conjugacy class graph of G is the graph whose vertices are the conjugacy classes of non-identity elements of G, two classes C and D adjacent if there exist $x \in C$ and $y \in D$ such that $\langle x, y \rangle$ is a solvable group.

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Given a positive integer k, there are only finitely many finite groups whose solvable conjugacy class graph has clique number k.

We do not have a good bound for the order of such a group.

Also, our proof uses the Classification of Finite Simple Groups, in a "light-touch" way; we do not know if this can be avoided.

Characterizing classes of groups

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I will give two examples of the second way. There are results on the first as well: for example, Pallabi Manna, Ranjit Mehatari and I studied groups whose power graph is a cograph (that is, contains no induced 4-vertex path); and Xuanlong Ma, Natalia Maslova and I have similar results for the commuting graph.

Power graph and enhanced power graph

These two graphs have as vertices the elements of G. The power graph of G, two vertices joined if one is a power of the other; in the enhanced power graph, two vertices are joined if both are powers of the same element. Thus, the power graph is a spanning subgraph of the enhanced power graph.

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The Gruenberg–Kegel graph of G has vertices the prime divisors of $|G|$, with an edge from p to q if G contains an element of order pq.

A group G is called an EPPO group if every element has prime power order. These were first investigated by Higman in the 1950s, who found the solvable EPPO groups; in the 1960s, Suzuki found the simple ones; and in 1981, Brandl found all these groups.

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Our proof, incidentally, resembles a classic alternating-paths argument from matching theory.

Super graphs on groups

In a paper with G. Arunkumar, Rajat Kanti Nath and Lavanya Selvaganesh, we proposed the following definition. If Γ is a type of graph defined on groups, then there is a super version of Γ, in which two elements x and y are joined if there exist conjugates x' and y' of x and y which are joined in Γ. (This is the conjugacy supergraph; a similar construction applies for other equivalence relations.)

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A Dedekind group is a group in which every subgroup is normal. Dedekind showed that such a finite group is either abelian, or of the form $Q \times A \times B$, where Q is the quaternion group of order 8, A an elementary abelian 2-group, and B an abelian group of odd order.

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Let G be a finite group. Then the power graph and super power graph of G are equal if and only if G is a Dedekind group. The same holds for the enhanced power graph and the super enhanced power graph.

Super graphs, 2

A group G is a 2-Engel group if it satisfies the identity $[x, y, y] = 1$ for all $x, y \in G$, where $[x, y]$ is the commutator $x^{-1}y^{-1}xy$, and $[x, y, z] = [[x, y], z]$.

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A nilpotent group of class 2 satisfies the identity $[x, y, z] = 1$ for all $x, y, z \in G$, so is obviously 2-Engel. In the other direction, Hopkins and Levi independently showed that a 2-Engel group is nilpotent of class 3, and is “close” to being nilpotent of class 2.

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Suppose that the commuting graph of G is equal to the conjugacy supercommuting graph. If x and y commute, then they are joined in the commuting graph, and hence in the supercommuting graph; thus every conjugate of y commutes with x. So $C_G(x)$ is a union of conjugacy classes, and hence is normal in G. By the above result, G is 2-Engel.

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The argument straightforwardly reverses.

Groups with isomorphic commuting graph

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Commutation in a group G is a map from $(G/Z(G)) \times (G/Z(G))$ to G'. Two groups G and H are isoclinic if there are isomorphisms $\alpha : G/Z(G) \to H/Z(H)$ and $\beta : G' \to H'$ so that $[g_1 Z(G)\alpha, g_2 Z(G)\alpha] = [g_1, g_2]\beta$.

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However, it fails for class 3. Let G be `SmallGroup(64,182)` in the GAP library. Then $G \times C_2$ has the same commuting graph as any Schur cover of G, although it is not isoclinic to any of them.

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- The clique number of the enhanced power graph of G is the largest order of an element of G. (A maximal clique in the enhanced power graph is a maximal cyclic subgroup.)
- Let $f(n)$ be the clique number of the power graph of a cyclic group of order n. Then the clique number of the power graph of G is the maximum value of $f(n)$ as n runs over all orders of elements of G.

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We have $f(n) \geq \phi(n)$, where ϕ is Euler's function. This is not too much smaller than n (it is bounded below by $cn/\log\log n$), so the clique numbers of the two graphs are not too far apart.

A small detour

I can't resist mentioning a cute result here. Let $f(n)$ be the clique number of the power graph of a cyclic group of order n. This is an arithmetic function of n, and was calculated by Alireza, Ahmad and Abbas. But there is a nice estimate for it:

$$\phi(n) \leq f(n) \leq c\phi(n),$$

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Another feature of these two graphs is that their complements contain graphs defined in terms of minimal generating sets for the group, the so-called independence graph and rank graph. I will not define these here. I will mention that Saul Freedman, Andrea Lucchini, Daniele Nemmi, and Colva Roney-Dougal have determined the groups for which the independence graph is the complement of the power graph, or the rank graph is the complement of the enhanced power graph.

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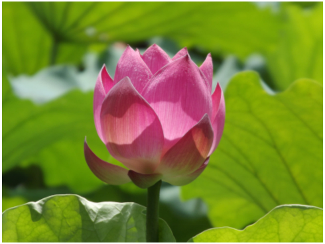

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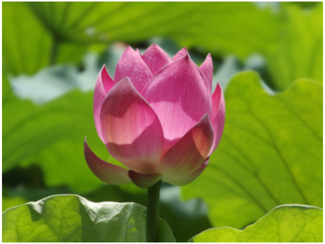

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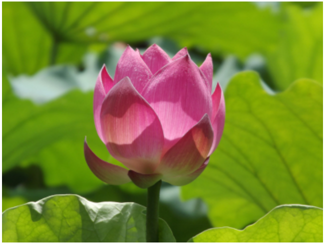

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Nearly ten years ago, Colva Roney-Dougal and I noticed that the automorphism group of the generating graph of A_5 (a group of order 60) has order 23482733690880. This impressively large group can almost all be stripped away.

Finding the jewel in the lotus, 2

Two vertices x and y of a graph Γ are twins if they have the same neighbours, apart possibly from one another. (Thus there are two kinds of twins; but this will not bother us.) Twin reduction is the process of repeatedly identifying twin vertices until no twins remain. It is not hard to show that the result of this process, up to isomorphism, does not depend on the order of the reductions. I will call this result the cokernel of Γ. Recall that Γ is a cograph if it contains no induced 4-vertex path.

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Proposition

The cokernel of Γ is the 1*-vertex graph if and only Γ is a cograph.*

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Proposition

The cokernel of Γ *is the* 1*-vertex graph if and only* Γ *is a cograph.*

Note that I have invented my own terminology here.

Finding the jewel in the lotus, 3

Graphs defined on groups tend to have many twins: if x has order greater than 2, then usually x and x^d are twins for any d coprime to the order of x. So we should apply twin reduction, and reach the cokernel of G.

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We applied this process to the difference graph $D(G)$ of a group G, the graph whose edges are the edges of the enhanced power graph which are not in the power graph. We expect this to be a fairly sparse graph and potentially to contain interesting stuff.

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At this stage, we are doing "experimental mathematics". Empirically, simple groups G seem to fall into four types, as on the next slide.

Finding the jewel in the lotus, 4

- Type 1: G is an EPPO group. Then $D(G)$ has no edges. The simple groups are a few $\mathrm{PSL}(2,q)$ and $\mathrm{Sz}(q)$ together with $\mathrm{PSL}(3,4)$.

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- Type 2: $D(G)$ is a cograph, so its cokernel has a single vertex. This includes some further $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$.

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- Type 3: The cokernel of $D(G)$ consists of a large number of isomorphic small graphs, e.g. 253 or 325 copies of $K_5 - P_4$ in $\mathrm{PSL}(2,23)$ and $\mathrm{PSL}(2,25)$ respectively.

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- Type 4: an interesting connected graph typically with large girth. I give a few examples on the next slide.

But I am sure there is much more to be found. Please try your hand!

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- Let $G = \mathrm{PSL}(3, 3)$. Then the cokernel of $D(G)$ is the following graph defined in the projective plane of order 3. The vertices are the ordered pairs (P, L) where P is a point and L a line. These are of two types: flags (where P and L are incident) and antiflags (where they are not). All edges join a flag to an antiflag; the antiflag (P, L) is incident with the flags (Q, M) where P is incident with M and Q with L. This graph has diameter 5 and girth 6, and 169 vertices.

- Let G be the Mathieu group M_{11}. Then the cokernel of $D(G)$ is a graph on 385 vertices; it is bipartite, with parts of size 165 and 220 (each of these sets an orbit of the automorphism group, which is just M_{11}), and has diameter 10 and girth 10. The valencies of vertices in the two parts are 4 and 3 respectively.

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- The (non-simple) Ree group $R_1(3) \cong \mathrm{P\Gamma L}(2,8)$ gives a semiregular bipartite graph on $63 + 84$ vertices, with valencies 4 and 3, with diameter 5 and girth 6.

Other graphs

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Sometimes we have to dig a little deeper. The same graph obtained from the difference graph of M_{11} above can also be obtained from the power graph, but we have to perform further reductions to get it.

… for your attention.