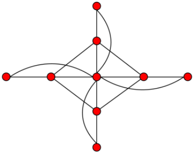
# An algebraist looks at discrete mathematics

Peter J. Cameron, University of St Andrews

ADMA Colloquium Lecture, 18 May 2024



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3. A beautiful graph from the Mathieu group $M_{11}$.

I can't say much about each of these but I hope to give you a little taste. The topics will be separated by pictures of bridges; these tell you that, if you have lost the thread, you can now pick up a new thread.

# Forth bridge, Scotland

# 1. Graph spectra and root systems

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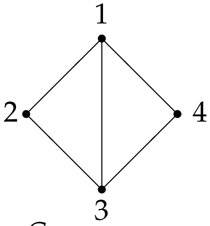
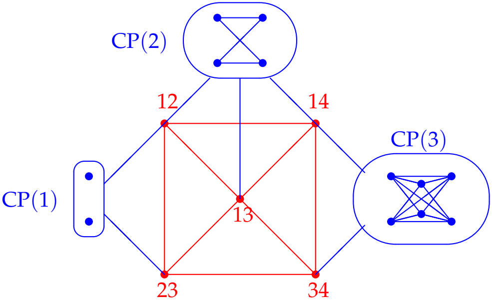
A generalized line graph is obtained from a line graph by attaching a cocktail party graph associated with each vertex. The next slide shows an example.

# A generalized line graph




$G$

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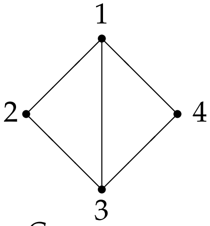


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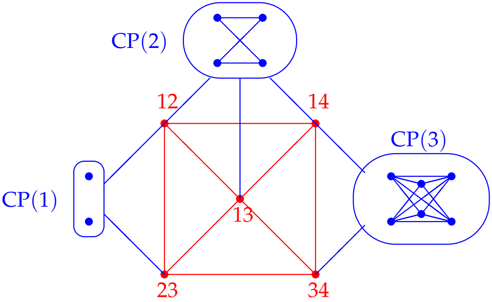


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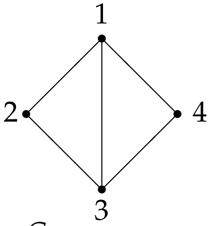


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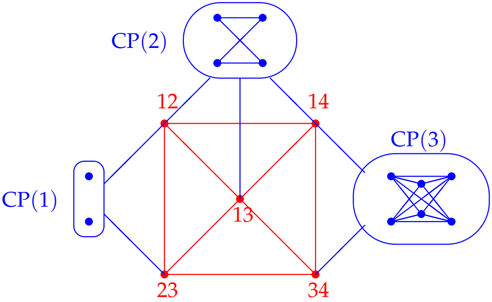


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Hoffman's conjecture is true in a strong sense. But the proof (by Jean-Marie Goethals, Jaap Seidel, Ernie Shult, and me) came from a completely different direction, using the theory of root systems.

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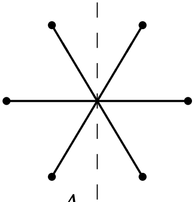
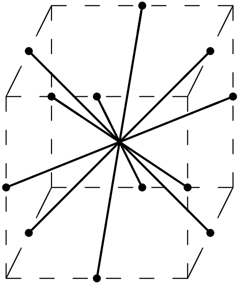


$A_2$



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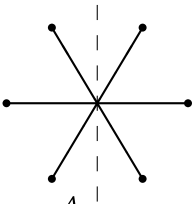

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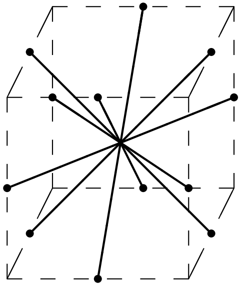

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They arose first in the theory of Lie algebras, but now occur everywhere from singularity theory to general relativity, from cluster algebras to finite simple groups.

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The important case for us is when all the vectors in $S$ have the same length. In this case, the angle between any two roots is $60^\circ$, $90^\circ$, $120^\circ$, or $180^\circ$.
We call the root system indecomposable if the space is not an orthogonal direct sum of two subspaces which contain all the vectors of $S$.

# The ADE classification

### Theorem

*An indecomposable root system with all roots of the same length is of one of the types $A_d$ ($d \geq 1$), $D_d$ ($d \geq 4$), $E_6$, $E_7$, $E_8$. (The subscript denotes the dimension.)*

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The systems are explicitly known and are easy to work with. These particular systems are ubiquitous in mathematics. A forthcoming book "ADE: Patterns in Mathematics" by Pierre Dechant, Yang He, John McKay and me traces some of their many occurrences in different parts of our subject.

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Here is a sketch proof. Let $A$ be the adjacency matrix. Then $2I + A$ is positive semi-definite, and so is the matrix of inner products of a set of vectors in a real vector space. These vectors all have length $\sqrt{2}$ and any two make angles of $90^\circ$ or $60^\circ$.

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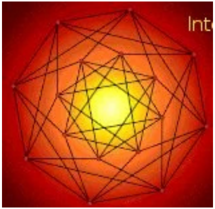


The Shrikhande graph (thanks to Ambat Vijayakumar)

# Howrah bridge, Kolkata

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The reason for this is even more surprising.

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The Erdős–Rényi proof is a nonconstructive existence proof: if something occurs with probability 1, it certainly occurs. But at almost the same time, Rado constructed a countable universal graph which turned out to be $R$. His construction was as follows. The vertex set is the set $\mathbb{N}$ of natural numbers (including 0). Given $x$ and $y$, with $x < y$, write $y$ in base 2: now join $x$ and $y$ if and only if the $x$-th digit of $y$ is 1.

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But even Fraïssé was not the first …

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He is known for several results in the foundations of topology, but the theorem I will talk about was less well known; it was published posthumously from the papers he left.
In 2000, I spoke about the random graph at the European Congress of Mathematics in Barcelona. After the talk, I was approached by someone who introduced himself as Anatoly Vershik, and asked if I knew about the Urysohn space. This is a Polish space (a complete second-countable metric space) which is universal and homogeneous.

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One can replace the rational numbers by other sets such as the non-negative integers (giving the integral Urysohn space, a distance-transitive graph) or the set $\{0, 1, 2\}$. Metric spaces with distances $\{0, 1, 2\}$ can be regarded as graphs (distance 1 is adjacency) and conversely. Now the $\{0, 1, 2\}$ Urysohn space is the random graph $R$!

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Vershik and I studied this and found various analogies between the random graph and the Urysohn space.

# 25 April bridge, Lisbon

# 3. Graphs on groups

In 1955, Brauer and Fowler published a paper which, with hindsight, was the first step towards the classification of the finite simple groups. They proved that there are only finitely many simple groups of even order containing an involution (an element of order 2) with a prescribed centralizer. The "even order" provision was needed because this predates the famous Feit–Thompson theorem that a non-abelian finite simple group must necessarily have even order.

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Though the word "graph" does not occur in the paper, the main tool they used was the commuting graph of the group, the graph whose vertices are the non-identity elements, two vertices $x$ and $y$ joined if they commute (that is, $xy = yx$).

## 3. Graphs on groups

In 1955, Brauer and Fowler published a paper which, with hindsight, was the first step towards the classification of the finite simple groups. They proved that there are only finitely many simple groups of even order containing an involution (an element of order 2) with a prescribed centralizer. The "even order" provision was needed because this predates the famous Feit–Thompson theorem that a non-abelian finite simple group must necessarily have even order.

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Since then, several more graphs on groups have been defined and studied, and many interactions found. (I must thank my friend Ambat Vijayakumar for organising a study group in 2021 which led to much of this research, as well as many colleagues, in India and elsewhere, who have collaborated with me on this.)

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It may happen that the power graph and enhanced power graph coincide, so that the difference graph is null. Groups with these properties are called EPPO groups, those in which every element has prime power order. They were determined by the efforts of Higman, Suzuki and Brandl.

# Twin reduction

Two vertices of a graph are twins if they have the same neighbours (possibly excepting each other). The process of twin reduction consists of repeatedly choosing a pair of twins and identifying them until no twins remain. The result of twin reduction is unique up to isomorphism, independent of the order of the process.

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It is an open problem which groups have the property that their difference graph is a cograph.

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- $\Gamma$ has diameter 10 and girth 10.
- The automorphism group of $\Gamma$ is the Mathieu group $M_{11}$.

# What next?

There are many groups, several graphs defined on groups, and a few different ways of "reducing" a graph in addition to twin reduction. So there is plenty more to explore. I invite you all to help explore this and find more beautiful graphs.

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But that is the end of my story.


… for your attention.