Topologies, filters and permutation groups

Peter J. Cameron, University of St Andrews

Celebrating 30 years of Dugald

MAC30, Leeds
October 2024

Dugald's sibs, Ambleside, 2007

Tunnel Mountain, Banff, 24 November 2014

Happy 30th, Dugald in Leeds!

Before moving to Leeds, Dugald was my student in Oxford, my host during a short visit to Simon Fraser (he took me up the Squamish Chief, where I got frostbite, since I had just come from a heatwave in southern California), and colleague at Queen Mary University of London.

Happy 30th, Dugald in Leeds!

Before moving to Leeds, Dugald was my student in Oxford, my host during a short visit to Simon Fraser (he took me up the Squamish Chief, where I got frostbite, since I had just come from a heatwave in southern California), and colleague at Queen Mary University of London.
I have always said that I reckon I've learned more from my students than they have from me.

Happy 30th, Dugald in Leeds!

Before moving to Leeds, Dugald was my student in Oxford, my host during a short visit to Simon Fraser (he took me up the Squamish Chief, where I got frostbite, since I had just come from a heatwave in southern California), and colleague at Queen Mary University of London.

I have always said that I reckon I've learned more from my students than they have from me.

Dugald is the best example of that. While he was a student, he produced some fiendishly clever arguments, and the only way I could understand them was to find my own proofs of some of his results.

Happy 30th, Dugald in Leeds!

Before moving to Leeds, Dugald was my student in Oxford, my host during a short visit to Simon Fraser (he took me up the Squamish Chief, where I got frostbite, since I had just come from a heatwave in southern California), and colleague at Queen Mary University of London.

I have always said that I reckon I've learned more from my students than they have from me.

Dugald is the best example of that. While he was a student, he produced some fiendishly clever arguments, and the only way I could understand them was to find my own proofs of some of his results.

This talk contains an example of the same process at work.

Topology in permutation groups

Some time in the 1980s, Helmut Wielandt gave a talk at Oberwolfach with this title, asking the question “What does it tell us about a permutation group if it is a group of homeomorphisms of a non-trivial topology?” (Here and below, “trivial” means “invariant under the symmetric group”.)

Topology in permutation groups

Some time in the 1980s, Helmut Wielandt gave a talk at Oberwolfach with this title, asking the question "What does it tell us about a permutation group if it is a group of homeomorphisms of a non-trivial topology?" (Here and below, "trivial" means "invariant under the symmetric group".)
His answer was "Not much". This was perhaps a bit pessimistic, and this lecture is also meant to throw a more positive light on Wielandt's question.

Topology in permutation groups

Some time in the 1980s, Helmut Wielandt gave a talk at Oberwolfach with this title, asking the question “What does it tell us about a permutation group if it is a group of homeomorphisms of a non-trivial topology?” (Here and below, “trivial” means “invariant under the symmetric group”.)
His answer was “Not much”. This was perhaps a bit pessimistic, and this lecture is also meant to throw a more positive light on Wielandt’s question.
To be clear, this does not refer to topology *of* permutation groups. Any permutation group carries a natural topology, that of pointwise convergence, which is very important in the theory of permutation groups and related parts of model theory.

Strong primitivity

Helmut Wielandt observed that the usual notion of primitivity is not strong enough to extend some results about finite permutation groups to the infinite.

Strong primitivity

Helmut Wielandt observed that the usual notion of primitivity is not strong enough to extend some results about finite permutation groups to the infinite.

For example, suppose that G is a primitive permutation group on the finite set Ω, and Δ a non-empty proper subset of Ω. Then for any $x, y \in \Omega$, there exists $g \in G$ such that $xg \in \Delta$ but $yg \notin \Delta$. But if $\Omega = \mathbb{Q}$, G is the group of order-preserving permutations of $\mathbb{Q}$, Δ the set of positive rationals, and $x > y$, then G is primitive but no such g exists.

Strong primitivity

Helmut Wielandt observed that the usual notion of primitivity is not strong enough to extend some results about finite permutation groups to the infinite.

For example, suppose that G is a primitive permutation group on the finite set Ω, and Δ a non-empty proper subset of Ω. Then for any $x, y \in \Omega$, there exists $g \in G$ such that $xg \in \Delta$ but $yg \notin \Delta$. But if $\Omega = \mathbb{Q}$, G is the group of order-preserving permutations of $\mathbb{Q}$, Δ the set of positive rationals, and $x > y$, then G is primitive but no such g exists.

To rectify this, the permutation group G is said to be strongly primitive if there is no non-trivial partial preorder (reflexive and transitive relation) preserved by G. Then the above result holds for strongly primitive groups. (I will usually omit the word "partial".)

Preorders and topologies

Thus, just as an imprimitive group preserves a nontrivial partition of Ω, a group which is not strongly primitive preserves a nontrivial partial preorder.

Preorders and topologies

Thus, just as an imprimitive group preserves a nontrivial partition of Ω, a group which is not strongly primitive preserves a nontrivial partial preorder.

Now there is a connection between topologies and preorders. Given a topology $\mathcal{T}$, define a preorder $\to$ by $x \to y$ if and only if every open set containing x contains y. In the other direction, given a preorder $\to$, define a topology where the sets $U_x = \{y : x \to y\}$ form a base.

Preorders and topologies

Thus, just as an imprimitive group preserves a nontrivial partition of Ω, a group which is not strongly primitive preserves a nontrivial partial preorder.

Now there is a connection between topologies and preorders. Given a topology $\mathcal{T}$, define a preorder $\to$ by $x \to y$ if and only if every open set containing x contains y. In the other direction, given a preorder $\to$, define a topology where the sets $U_x = \{y : x \to y\}$ form a base.

The map preorder $\to$ topology $\to$ preorder is always the identity, but topology $\to$ preorder $\to$ topology gives a stronger topology. We say that a topology is relational if it is fixed by this map.

Preorders and topologies

Thus, just as an imprimitive group preserves a nontrivial partition of Ω, a group which is not strongly primitive preserves a nontrivial partial preorder.

Now there is a connection between topologies and preorders. Given a topology $\mathcal{T}$, define a preorder $\rightarrow$ by $x \rightarrow y$ if and only if every open set containing x contains y. In the other direction, given a preorder $\rightarrow$, define a topology where the sets $U_x = \{y : x \rightarrow y\}$ form a base.

The map preorder $\rightarrow$ topology $\rightarrow$ preorder is always the identity, but topology $\rightarrow$ preorder $\rightarrow$ topology gives a stronger topology. We say that a topology is relational if it is fixed by this map.

Thus, for example, every finite topology is relational.

Preorders and topologies

Thus, just as an imprimitive group preserves a nontrivial partition of Ω, a group which is not strongly primitive preserves a nontrivial partial preorder.

Now there is a connection between topologies and preorders. Given a topology $\mathcal{T}$, define a preorder $\rightarrow$ by $x \rightarrow y$ if and only if every open set containing x contains y. In the other direction, given a preorder $\rightarrow$, define a topology where the sets $U_x = \{y : x \rightarrow y\}$ form a base.

The map preorder $\rightarrow$ topology $\rightarrow$ preorder is always the identity, but topology $\rightarrow$ preorder $\rightarrow$ topology gives a stronger topology. We say that a topology is relational if it is fixed by this map.

Thus, for example, every finite topology is relational.

Fraïssé's Theorem guarantees the existence of a countable homogeneous preorder.

Primitivity and topology

Theorem

Let G be a transitive permutation group on Ω.

Primitivity and topology

Theorem

Let G be a transitive permutation group on Ω.

- *G is primitive if and only if every non-trivial G-invariant topology is T0.*

Primitivity and topology

Theorem

Let G be a transitive permutation group on Ω.

- *G is primitive if and only if every non-trivial G-invariant topology is T0.*
- *G is strongly primitive if and only if every non-trivial G-invariant topology is T1.*

Primitivity and topology

Theorem

Let G be a transitive permutation group on Ω.

- *G is primitive if and only if every non-trivial G-invariant topology is T0.*
- *G is strongly primitive if and only if every non-trivial G-invariant topology is T1.*

A couple of remarks:

Primitivity and topology

Theorem

Let G be a transitive permutation group on Ω.

- *G is primitive if and only if every non-trivial G-invariant topology is T0.*
- *G is strongly primitive if and only if every non-trivial G-invariant topology is T1.*

A couple of remarks:

- G preserves a non-T0 (resp. non-T1) topology if and only if it preserves a relational topology with the same property.

Primitivity and topology

Theorem

Let G be a transitive permutation group on Ω.

- *G is primitive if and only if every non-trivial G-invariant topology is T0.*
- *G is strongly primitive if and only if every non-trivial G-invariant topology is T1.*

A couple of remarks:

- G preserves a non-T0 (resp. non-T1) topology if and only if it preserves a relational topology with the same property.
- The topology derived from the universal homogeneous preorder (resp. partial order) is non-T0 (resp. non-T1).

Primitivity and topology

Theorem

Let G be a transitive permutation group on Ω.

- *G is primitive if and only if every non-trivial G-invariant topology is T0.*
- *G is strongly primitive if and only if every non-trivial G-invariant topology is T1.*

A couple of remarks:

- G preserves a non-T0 (resp. non-T1) topology if and only if it preserves a relational topology with the same property.
- The topology derived from the universal homogeneous preorder (resp. partial order) is non-T0 (resp. non-T1).

Question

What about higher separation axioms?

Filters

A filter $\mathcal{F}$ on Ω is a family of sets which is closed upwards and closed under pairwise intersection. Thus, a filter containing two disjoint sets is the power set of Ω. A filter is non-trivial if this is not the case.

Filters

A filter $\mathcal{F}$ on Ω is a family of sets which is closed upwards and closed under pairwise intersection. Thus, a filter containing two disjoint sets is the power set of Ω. A filter is non-trivial if this is not the case.

An ultrafilter is a maximal non-trivial filter. A filter is principal if it consists of all sets containing a fixed set F. Thus a principal ultrafilter consists of all sets containing a fixed point $x \in \Omega$.

Filters

A filter $\mathcal{F}$ on Ω is a family of sets which is closed upwards and closed under pairwise intersection. Thus, a filter containing two disjoint sets is the power set of Ω. A filter is non-trivial if this is not the case.
An ultrafilter is a maximal non-trivial filter. A filter is principal if it consists of all sets containing a fixed set F. Thus a principal ultrafilter consists of all sets containing a fixed point $x \in \Omega$.

Proposition

A non-principal ultrafilter contains the filter of all cofinite sets.

Filters

A filter $\mathcal{F}$ on Ω is a family of sets which is closed upwards and closed under pairwise intersection. Thus, a filter containing two disjoint sets is the power set of Ω. A filter is non-trivial if this is not the case.
An ultrafilter is a maximal non-trivial filter. A filter is principal if it consists of all sets containing a fixed set F. Thus a principal ultrafilter consists of all sets containing a fixed point $x \in \Omega$.

Proposition

A non-principal ultrafilter contains the filter of all cofinite sets.

Zorn's Lemma guarantees that non-principal ultrafilters exist; their automorphism groups are maximal subgroups of the symmetric group.

Filters

A filter $\mathcal{F}$ on Ω is a family of sets which is closed upwards and closed under pairwise intersection. Thus, a filter containing two disjoint sets is the power set of Ω. A filter is non-trivial if this is not the case.

An ultrafilter is a maximal non-trivial filter. A filter is principal if it consists of all sets containing a fixed set F. Thus a principal ultrafilter consists of all sets containing a fixed point $x \in \Omega$.

Proposition

A non-principal ultrafilter contains the filter of all cofinite sets.

Zorn's Lemma guarantees that non-principal ultrafilters exist; their automorphism groups are maximal subgroups of the symmetric group.

Note that, if $\mathcal{F}$ is a filter, then $\mathcal{F} \cup \{\emptyset\}$ is a topology, and is non-trivial if $\mathcal{F}$ is.

Maximal subgroups

One of the most important things we want to know about a finite group is its list of subgroups. For finite symmetric groups, this has been a very active area of research, centred round the O'Nan–Scott Theorem.

Maximal subgroups

One of the most important things we want to know about a finite group is its list of subgroups. For finite symmetric groups, this has been a very active area of research, centred round the O'Nan–Scott Theorem.

Infinite groups are more problematic, since they may have subgroups not contained in maximal subgroups. For example, the symmetric group of countable degree is uncountable; any countable subgroup is contained in a larger countable subgroup, so cannot be maximal.

Maximal subgroups

One of the most important things we want to know about a finite group is its list of subgroups. For finite symmetric groups, this has been a very active area of research, centred round the O'Nan–Scott Theorem.

Infinite groups are more problematic, since they may have subgroups not contained in maximal subgroups. For example, the symmetric group of countable degree is uncountable; any countable subgroup is contained in a larger countable subgroup, so cannot be maximal.

In 1990, Dugald Macpherson and Cheryl Praeger proved:

Maximal subgroups

One of the most important things we want to know about a finite group is its list of subgroups. For finite symmetric groups, this has been a very active area of research, centred round the O'Nan–Scott Theorem.

Infinite groups are more problematic, since they may have subgroups not contained in maximal subgroups. For example, the symmetric group of countable degree is uncountable; any countable subgroup is contained in a larger countable subgroup, so cannot be maximal.

In 1990, Dugald Macpherson and Cheryl Praeger proved:

Theorem

A permutation group of countable degree which is not highly transitive is contained in a maximal subgroup of $\mathrm{Sym}(\Omega)$.

Maximal subgroups

One of the most important things we want to know about a finite group is its list of subgroups. For finite symmetric groups, this has been a very active area of research, centred round the O'Nan–Scott Theorem.

Infinite groups are more problematic, since they may have subgroups not contained in maximal subgroups. For example, the symmetric group of countable degree is uncountable; any countable subgroup is contained in a larger countable subgroup, so cannot be maximal.

In 1990, Dugald Macpherson and Cheryl Praeger proved:

Theorem

A permutation group of countable degree which is not highly transitive is contained in a maximal subgroup of $\mathrm{Sym}(\Omega)$.

(Highly transitive means transitive on n-tuples of distinct points for all n.)

The proof

The proof involves three steps. Let G be a group on a countable set Ω which is not highly transitive.

The proof

The proof involves three steps. Let G be a group on a countable set Ω which is not highly transitive.

- G preserves a topology on Ω.

The proof

The proof involves three steps. Let G be a group on a countable set Ω which is not highly transitive.

- G preserves a topology on Ω.
- G preserves a filter on Ω.

The proof

The proof involves three steps. Let G be a group on a countable set Ω which is not highly transitive.

- G preserves a topology on Ω.
- G preserves a filter on Ω.
- G is contained in a maximal subgroup of $\mathrm{Sym}(\Omega)$.

The proof

The proof involves three steps. Let G be a group on a countable set Ω which is not highly transitive.

- G preserves a topology on Ω.
- G preserves a filter on Ω.
- G is contained in a maximal subgroup of $\mathrm{Sym}(\Omega)$.

The argument in their paper actually uses ideals rather than filters (an ideal is a family of sets closed downwards and under pairwise union, so the complements of the sets in a filter form an ideal and *vice versa*).

Topologies and filters

Note that, if $\mathcal{F}$ is a filter, then $\mathcal{F} \cup \{\emptyset\}$ is a topology. The other direction is less trivial; Macpherson and Praeger use substantial machinery from model theory (the theorems of Ehrenfeucht–Mostowski, Engeler–Ryll-Nardzewski–Svenonius, and Cherlin–Harrington–Lachlan).

Topologies and filters

Note that, if $\mathcal{F}$ is a filter, then $\mathcal{F} \cup \{\emptyset\}$ is a topology. The other direction is less trivial; Macpherson and Praeger use substantial machinery from model theory (the theorems of Ehrenfeucht–Mostowski, Engeler–Ryll-Nardzewski–Svenonius, and Cherlin–Harrington–Lachlan).
Here is a more direct argument, producing the filter directly from the topology. Only two constructions of filters are required.

Topologies and filters

Note that, if $\mathcal{F}$ is a filter, then $\mathcal{F} \cup \{\emptyset\}$ is a topology. The other direction is less trivial; Macpherson and Praeger use substantial machinery from model theory (the theorems of Ehrenfeucht–Mostowski, Engeler–Ryll-Nardzewski–Svenonius, and Cherlin–Harrington–Lachlan).

Here is a more direct argument, producing the filter directly from the topology. Only two constructions of filters are required.

Theorem

A primitive group on a countable set which preserves a non-trivial topology preserves a non-trivial filter.

A preliminary result

A moiety in a countable set is an infinite and co-infinite subset.

A preliminary result

A moiety in a countable set is an infinite and co-infinite subset.

Proposition

Let G be a primitive permutation group on the countable set Ω.

A preliminary result

A moiety in a countable set is an infinite and co-infinite subset.

Proposition

Let G be a primitive permutation group on the countable set Ω.

(a) *G preserves a non-trivial topology if and only if there is a moeity Δ such that the intersection of any finite number of images of Δ under G is empty or infinite.*

A preliminary result

A moiety in a countable set is an infinite and co-infinite subset.

Proposition

Let G be a primitive permutation group on the countable set Ω.

(a) *G preserves a non-trivial topology if and only if there is a moeity Δ such that the intersection of any finite number of images of Δ under G is empty or infinite.*

(b) *G preserves a non-trivial filter if and only if there is a moeity Δ such that the intersection of any finite number of images of Δ under G is infinite.*

Proof

For (a), show using primitivity that if there is a finite non-empty open set then the topology is discrete, so trivial; otherwise a non-cofinite open set Δ satisfies the condition.

Proof

For (a), show using primitivity that if there is a finite non-empty open set then the topology is discrete, so trivial; otherwise a non-cofinite open set Δ satisfies the condition. Conversely, if Δ satisfies the condition, then the non-empty intersections of finitely many translates of Δ form a basis for a topology.

Proof

For (a), show using primitivity that if there is a finite non-empty open set then the topology is discrete, so trivial; otherwise a non-cofinite open set Δ satisfies the condition. Conversely, if Δ satisfies the condition, then the non-empty intersections of finitely many translates of Δ form a basis for a topology.

For (b), we use Neumann's lemma: if G has no finite orbits on Ω and A, B are finite sets, then there exists $g \in G$ such that $Ag \cap B = \emptyset$. So a non-trivial filter admitting a transitive group contains no finite sets, and any set of the filter has the required property.

Proof

For (a), show using primitivity that if there is a finite non-empty open set then the topology is discrete, so trivial; otherwise a non-cofinite open set Δ satisfies the condition.
Conversely, if Δ satisfies the condition, then the non-empty intersections of finitely many translates of Δ form a basis for a topology.
For (b), we use Neumann's lemma: if G has no finite orbits on Ω and A, B are finite sets, then there exists $g \in G$ such that $Ag \cap B = \emptyset$. So a non-trivial filter admitting a transitive group contains no finite sets, and any set of the filter has the required property.
Conversely, if Δ satisfies the condition, then the sets containing intersections of finitely many translates of Δ is a filter.

Proof of the theorem

Each of the following is a filter:

Proof of the theorem

Each of the following is a filter:

- the sets containing finite intersections of dense open sets;

Proof of the theorem

Each of the following is a filter:

- the sets containing finite intersections of dense open sets;
- the complements of finite unions of discrete sets.

Proof of the theorem

Each of the following is a filter:

- the sets containing finite intersections of dense open sets;
- the complements of finite unions of discrete sets.

So each of these families must be trivial: thus, every dense open set is cofinite, and every discrete set is finite.

Proof of the theorem

Each of the following is a filter:

- the sets containing finite intersections of dense open sets;
- the complements of finite unions of discrete sets.

So each of these families must be trivial: thus, every dense open set is cofinite, and every discrete set is finite.

Form a graph on Ω by joining x and y whenever there exist disjoint open sets containing these points.

Proof of the theorem

Each of the following is a filter:

- the sets containing finite intersections of dense open sets;
- the complements of finite unions of discrete sets.

So each of these families must be trivial: thus, every dense open set is cofinite, and every discrete set is finite.

Form a graph on Ω by joining x and y whenever there exist disjoint open sets containing these points.

If the graph has no infinite clique, then show that it is complete multipartite, contradicting primitivity unless it is null, in which case any two non-empty open sets intersect, and the sets containing non-empty open sets form a non-trivial filter.

Proof of the theorem

Each of the following is a filter:

- the sets containing finite intersections of dense open sets;
- the complements of finite unions of discrete sets.

So each of these families must be trivial: thus, every dense open set is cofinite, and every discrete set is finite.
Form a graph on Ω by joining x and y whenever there exist disjoint open sets containing these points.
If the graph has no infinite clique, then show that it is complete multipartite, contradicting primitivity unless it is null, in which case any two non-empty open sets intersect, and the sets containing non-empty open sets form a non-trivial filter.
If the graph has an infinite clique C, then the induced topology on C is Hausdorff, and so contains an infinite discrete set (an exercise in Sierpiński's book), a contradiction.

The random graph

There is more to say about filters, but first a digression, to introduce one of my favourite objects, the (Erdős–Rényi) countable random graph (aka the Rado graph). This is the graph obtained almost surely if edges on a countable vertex set are chosen by tossing a coin for each pair of vertices.

The random graph

There is more to say about filters, but first a digression, to introduce one of my favourite objects, the (Erdős–Rényi) countable random graph (aka the Rado graph). This is the graph obtained almost surely if edges on a countable vertex set are chosen by tossing a coin for each pair of vertices.
(This fact constitutes a valid non-constructive existence proof. Rado gave the first explicit construction of it.)

The random graph

There is more to say about filters, but first a digression, to introduce one of my favourite objects, the (Erdős–Rényi) countable random graph (aka the Rado graph). This is the graph obtained almost surely if edges on a countable vertex set are chosen by tossing a coin for each pair of vertices.
(This fact constitutes a valid non-constructive existence proof. Rado gave the first explicit construction of it.)
In case you were wondering, the coin does not have to be fair; we can even allow it to slowly become more biased as the construction proceeds.

Some facts about the countable random graph:

- it is universal (contains all finite and countable graphs as induced subgraphs) and homogeneous (any isomorphism between finite induced subgraphs extends to an automorphism);

Some facts about the countable random graph:

- it is universal (contains all finite and countable graphs as induced subgraphs) and homogeneous (any isomorphism between finite induced subgraphs extends to an automorphism);
- it is the Fraïssé limit of the class of finite graphs;

Some facts about the countable random graph:

- it is universal (contains all finite and countable graphs as induced subgraphs) and homogeneous (any isomorphism between finite induced subgraphs extends to an automorphism);
- it is the Fraïssé limit of the class of finite graphs;
- it is obtained by "undirecting" the membership relation in any countable model of ZF;

Some facts about the countable random graph:

- it is universal (contains all finite and countable graphs as induced subgraphs) and homogeneous (any isomorphism between finite induced subgraphs extends to an automorphism);
- it is the Fraïssé limit of the class of finite graphs;
- it is obtained by "undirecting" the membership relation in any countable model of ZF;
- it is obtained by taking the vertices to be the primes congruent to 1 mod 4, p and q joined if p is a quadratic residue mod q;

Some facts about the countable random graph:

- it is universal (contains all finite and countable graphs as induced subgraphs) and homogeneous (any isomorphism between finite induced subgraphs extends to an automorphism);
- it is the Fraïssé limit of the class of finite graphs;
- it is obtained by “undirecting” the membership relation in any countable model of ZF;
- it is obtained by taking the vertices to be the primes congruent to 1 mod 4, p and q joined if p is a quadratic residue mod q;
- its first-order theory is the theory of almost all finite graphs.

Some facts about the countable random graph:

- it is universal (contains all finite and countable graphs as induced subgraphs) and homogeneous (any isomorphism between finite induced subgraphs extends to an automorphism);
- it is the Fraïssé limit of the class of finite graphs;
- it is obtained by "undirecting" the membership relation in any countable model of ZF;
- it is obtained by taking the vertices to be the primes congruent to 1 mod 4, p and q joined if p is a quadratic residue mod q;
- its first-order theory is the theory of almost all finite graphs.

Rado's construction is equivalent to taking the unique model of hereditarily finite set theory and undirecting membership. (Most of ZF is not necessary for the third point above; the Axiom of Foundation is the crucial one.)

Generating filters

Given a family $\mathcal{A}$ of subsets of V, the filter generated by $\mathcal{A}$ is the set

$$\mathcal{F} = \{X \subseteq V : (\exists A_1, \ldots, A_n \in \mathcal{A})(A_1 \cap \cdots \cap A_n) \subseteq X\}.$$

Generating filters

Given a family $\mathcal{A}$ of subsets of V, the filter generated by $\mathcal{A}$ is the set

$$\mathcal{F} = \{X \subseteq V : (\exists A_1, \ldots, A_n \in \mathcal{A})(A_1 \cap \cdots \cap A_n) \subseteq X\}.$$

Two families $\mathcal{A}_1$ and $\mathcal{A}_2$ generate the same filter if and only if each member in $\mathcal{A}_2$ lies in the filter generated by $\mathcal{A}_1$ (that is, contains a finite intersection of sets of $\mathcal{A}_1$) and *vice versa*.

Neighbourhood filters

Let Γ be a graph on a countable vertex set V. We define the neighbourhood filter of Γ to be the filter generated by $\{\Gamma(v) : v \in V\}$, where $\Gamma(v)$ denotes the neighbourhood of v in Γ, the set of vertices adjacent to v.

Proposition

Suppose that Γ has the property that each vertex has a non-neighbour. Then the filter generated by the closed neighbourhoods $\overline{\Gamma}(v) = \Gamma(v) \cup \{v\}$ is equal to $\mathcal{F}_\Gamma$.

Neighbourhood filters

Let Γ be a graph on a countable vertex set V. We define the neighbourhood filter of Γ to be the filter generated by $\{\Gamma(v) : v \in V\}$, where $\Gamma(v)$ denotes the neighbourhood of v in Γ, the set of vertices adjacent to v.

Proposition

Suppose that Γ has the property that each vertex has a non-neighbour. Then the filter generated by the closed neighbourhoods $\overline{\Gamma}(v) = \Gamma(v) \cup \{v\}$ is equal to $\mathcal{F}_\Gamma$.

For we have $\Gamma(v) \subseteq \overline{\Gamma}(v)$, and, if w is not adjacent to v, then $\overline{\Gamma}(v) \cap \overline{\Gamma}(w) \subseteq \Gamma(v)$.

Neighbourhood filters

Let Γ be a graph on a countable vertex set V. We define the neighbourhood filter of Γ to be the filter generated by $\{\Gamma(v) : v \in V\}$, where $\Gamma(v)$ denotes the neighbourhood of v in Γ, the set of vertices adjacent to v.

Proposition

Suppose that Γ has the property that each vertex has a non-neighbour. Then the filter generated by the closed neighbourhoods $\overline{\Gamma}(v) = \Gamma(v) \cup \{v\}$ is equal to $\mathcal{F}_\Gamma$.

For we have $\Gamma(v) \subseteq \overline{\Gamma}(v)$, and, if w is not adjacent to v, then $\overline{\Gamma}(v) \cap \overline{\Gamma}(w) \subseteq \Gamma(v)$.
The condition on Γ is necessary. If Γ is the complete graph, the closed neighbourhoods generate the filter $\{V\}$, while the open neighbourhoods generate the filter of cofinite subsets of V.

Neighbourhood filter of the random graph

Let R denote the countable random graph.

Neighbourhood filter of the random graph

Let R denote the countable random graph.

Proposition

The following three conditions on a graph Γ *are equivalent:*

Neighbourhood filter of the random graph

Let R denote the countable random graph.

Proposition

The following three conditions on a graph Γ *are equivalent:*

(a) $\mathcal{F}_\Gamma$ *is nontrivial;*

Neighbourhood filter of the random graph

Let R denote the countable random graph.

Proposition

The following three conditions on a graph Γ *are equivalent:*

(a) $\mathcal{F}_\Gamma$ *is nontrivial;*

(b) Γ *contains* R *as a spanning subgraph;*

Neighbourhood filter of the random graph

Let R denote the countable random graph.

Proposition

The following three conditions on a graph Γ *are equivalent:*

(a) $\mathcal{F}_\Gamma$ *is nontrivial;*

(b) Γ *contains* R *as a spanning subgraph;*

(c) $\mathcal{F}_\Gamma \subseteq \mathcal{F}_R$.

Neighbourhood filter of the random graph

Let R denote the countable random graph.

Proposition

The following three conditions on a graph Γ *are equivalent:*

(a) $\mathcal{F}_\Gamma$ *is nontrivial;*

(b) Γ *contains* R *as a spanning subgraph;*

(c) $\mathcal{F}_\Gamma \subseteq \mathcal{F}_R$.

A filter is trivial if and only if it contains the empty set. So $\mathcal{F}_\Gamma$ is non-trivial if and only if any finitely many neighbourhoods have non-empty intersection. This is equivalent to the statement that R is a spanning subgraph of Γ. So (a) and (b) are equivalent.

Neighbourhood filter of the random graph

Let R denote the countable random graph.

Proposition

The following three conditions on a graph Γ *are equivalent:*

(a) $\mathcal{F}_\Gamma$ *is nontrivial;*

(b) Γ *contains* R *as a spanning subgraph;*

(c) $\mathcal{F}_\Gamma \subseteq \mathcal{F}_R$.

A filter is trivial if and only if it contains the empty set. So $\mathcal{F}_\Gamma$ is non-trivial if and only if any finitely many neighbourhoods have non-empty intersection. This is equivalent to the statement that R is a spanning subgraph of Γ. So (a) and (b) are equivalent.

If Γ contains R as a spanning subgraph, then $R(v) \subseteq \Gamma(v)$ for all v. So (b) implies (c). Conversely, $\mathcal{F}_R$ is non-trivial (by our proof that (b) implies (a)), so (c) implies (a).

Remark

This result shows that $\mathcal{F}_R$ is the unique maximal neighbourhood filter. But this uniqueness is only up to isomorphism. So part (c) really means that $\mathcal{F}_\Gamma$ is contained in a filter isomorphic to $\mathcal{F}_R$.

Remark

This result shows that $\mathcal{F}_R$ is the unique maximal neighbourhood filter. But this uniqueness is only up to isomorphism. So part (c) really means that $\mathcal{F}_\Gamma$ is contained in a filter isomorphic to $\mathcal{F}_R$.

For example, let T be the random 3-colouring of the edges of the complete graph, with colours red, green and blue. Let R_1 be the graph consisting of red edges, and R_2 the graph consisting of red and green edges, in T. Then both R_1 and R_2 are isomorphic to R. Since $R_1(v) \subseteq R_2(v)$, we have $\mathcal{F}_{R_2} \subseteq \mathcal{F}_{R_1}$. We show that the inequality is strict.

Remark

This result shows that $\mathcal{F}_R$ is the unique maximal neighbourhood filter. But this uniqueness is only up to isomorphism. So part (c) really means that $\mathcal{F}_\Gamma$ is contained in a filter isomorphic to $\mathcal{F}_R$.

For example, let T be the random 3-colouring of the edges of the complete graph, with colours red, green and blue. Let R_1 be the graph consisting of red edges, and R_2 the graph consisting of red and green edges, in T. Then both R_1 and R_2 are isomorphic to R. Since $R_1(v) \subseteq R_2(v)$, we have $\mathcal{F}_{R_2} \subseteq \mathcal{F}_{R_1}$. We show that the inequality is strict.

The set $R_1(v)$ belongs to $\mathcal{F}_{R_1}$. Suppose that it belongs to $\mathcal{F}_{R_2}$. Then there are vertices $w_1, \ldots, w_n$ such that

$$R_2(w_1) \cap \ldots \cap R_2(w_n) \subseteq R_1(v).$$

But, since the green graph is isomorphic to R, there is a vertex x joined to all of $v, w_1, \ldots, w_n$ by green edges; then x belongs to the left but not to the right, a contradiction.

Remark

This result shows that $\mathcal{F}_R$ is the unique maximal neighbourhood filter. But this uniqueness is only up to isomorphism. So part (c) really means that $\mathcal{F}_\Gamma$ is contained in a filter isomorphic to $\mathcal{F}_R$.

For example, let T be the random 3-colouring of the edges of the complete graph, with colours red, green and blue. Let R_1 be the graph consisting of red edges, and R_2 the graph consisting of red and green edges, in T. Then both R_1 and R_2 are isomorphic to R. Since $R_1(v) \subseteq R_2(v)$, we have $\mathcal{F}_{R_2} \subseteq \mathcal{F}_{R_1}$. We show that the inequality is strict.

The set $R_1(v)$ belongs to $\mathcal{F}_{R_1}$. Suppose that it belongs to $\mathcal{F}_{R_2}$. Then there are vertices $w_1, \ldots, w_n$ such that

$$R_2(w_1) \cap \ldots \cap R_2(w_n) \subseteq R_1(v).$$

But, since the green graph is isomorphic to R, there is a vertex x joined to all of $v, w_1, \ldots, w_n$ by green edges; then x belongs to the left but not to the right, a contradiction.

Similarly there are countable chains of filters isomorphic to $\mathcal{F}_R$.

Topologies

We get two topologies $\mathcal{T}_o$, $\mathcal{T}_c$ on the vertex set of the random graph by taking a sub-basis for the open sets to be the open, resp. closed, vertex neighbourhoods; that is, the open sets are all unions of finite intersections of open, resp. closed, neighbourhoods.

Topologies

We get two topologies $\mathcal{T}_o$, $\mathcal{T}_c$ on the vertex set of the random graph by taking a sub-basis for the open sets to be the open, resp. closed, vertex neighbourhoods; that is, the open sets are all unions of finite intersections of open, resp. closed, neighbourhoods.

Proposition

- *The families of open and closed neighbourhoods in R are isomorphic.*

Topologies

We get two topologies $\mathcal{T}_o$, $\mathcal{T}_c$ on the vertex set of the random graph by taking a sub-basis for the open sets to be the open, resp. closed, vertex neighbourhoods; that is, the open sets are all unions of finite intersections of open, resp. closed, neighbourhoods.

Proposition

- *The families of open and closed neighbourhoods in R are isomorphic.*
- *The topologies $\mathcal{T}_o$ and $\mathcal{T}_c$ are homeomorphic.*

Topologies

We get two topologies $\mathcal{T}_o$, $\mathcal{T}_c$ on the vertex set of the random graph by taking a sub-basis for the open sets to be the open, resp. closed, vertex neighbourhoods; that is, the open sets are all unions of finite intersections of open, resp. closed, neighbourhoods.

Proposition

- *The families of open and closed neighbourhoods in R are isomorphic.*
- *The topologies $\mathcal{T}_o$ and $\mathcal{T}_c$ are homeomorphic.*

To prove this, show that the Levi graphs of the two families of sets are both isomorphic to the generic bipartite graph, arising as the Fraïssé limit of of the class of all finite graphs with bipartition. (Bipartite graphs do not form a Fraïssé class but if we include a bipartition as part of the structure they do.)

Properties

These topologies are not Hausdorff but do have the T1 separation property. (Given two distinct vertices, there is a vertex joined to one but not the other.)

Properties

These topologies are not Hausdorff but do have the T1 separation property. (Given two distinct vertices, there is a vertex joined to one but not the other.)
Indeed, the automorphism group of R is strongly primitive, in Wielandt's sense, so any $\mathrm{Aut}(R)$-invariant topology must be T1.

Properties

These topologies are not Hausdorff but do have the T1 separation property. (Given two distinct vertices, there is a vertex joined to one but not the other.)

Indeed, the automorphism group of R is strongly primitive, in Wielandt's sense, so any $\mathrm{Aut}(R)$-invariant topology must be T1.

The homeomorphism groups are highly transitive.

Properties

These topologies are not Hausdorff but do have the T1 separation property. (Given two distinct vertices, there is a vertex joined to one but not the other.)
Indeed, the automorphism group of R is strongly primitive, in Wielandt's sense, so any $\mathrm{Aut}(R)$-invariant topology must be T1.
The homeomorphism groups are highly transitive.
Not much else is known. There is a very rich collection of highly transitive overgroups of the automorphism group of R, and certainly much more remains to be discovered.

… for your attention.

… for your attention.

Dugald, we wish you many more years of mathematics, mountaineering, etc.!