Graphs defined on groups: some interactions

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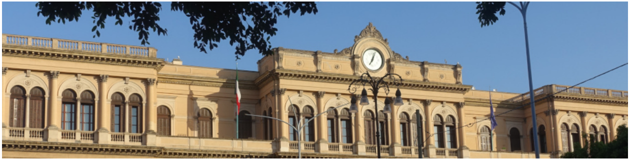

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Yesterday Mariagrazia Bianchi gave a talk on "Groups and graphs: recent results". I could have given my talk the same title, but there will be little in common between our talks.

My topic is one which has seen a huge upsurge in activity in the last decade or so. This concerns graphs whose vertex set is a group, defined by purely group-theoretic properties. The classic example is the commuting graph of a group, where the vertices are the elements of G and two vertices are joined if they commute.

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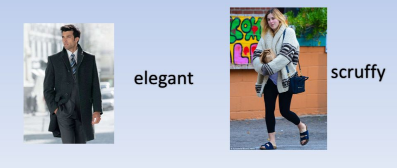

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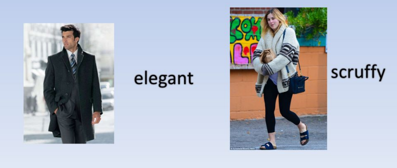

We would expect to find that graphs associated with algebraic structures are less scruffy than general graphs. Later I will show you some examples of beautiful graphs from groups.

An example

Here is a brief example, mentioned by Daniele Nemmi yesterday. The generating graph of a group has vertex set the non-identity group elements, two elements x and y joined if $\langle x, y \rangle = G$. Now not every group can be generated by two elements; but the Classification of Finite Simple Groups shows that every finite simple group is 2-generated.

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I think the term "spread" was introduced into graph theory by group theorists.

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There has been a lot of work on computing many different graph-theoretic parameters of some of the graphs. This is important work, but I regard it as more in the nature of filling in detail in the background of the picture.

New results about groups

The classic example of this is the 1955 paper by Brauer and Fowler in which they showed that there are only finitely many finite simple groups of even order which have a given involution centralizer. With hindsight, this was the first step in the thousand-mile journey to the Classification of the Finite Simple Groups. Their proof involved bounding the diameter of the commuting graph of such a group.

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Subsequent authors have bounded the order of such groups; but our extension goes in a different direction.

The soluble conjugacy class graph

Let G be a finite group. The soluble conjugacy class graph of G is the graph whose vertices are the conjugacy classes of non-identity elements of G, two classes C and D adjacent if there exist $x \in C$ and $y \in D$ such that $\langle x, y \rangle$ is a soluble group.

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We do not have a good bound for the order of such a group. Also, our proof uses the Classification of Finite Simple Groups, in a "light-touch" way; we do not know if this can be avoided. And also, we think it might be true if "soluble" is replaced by "nilpotent" or even "abelian".

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- Choose two different types of graphs defined on groups, and ask for which groups these two graphs coincide.

If we can classify the groups for which two types of graphs are equal, it is interesting to study the difference of the two graphs for other groups. I will show an example of this as well.

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An example of the second type involves the independence graph (two elements joined if they are contained in a minimal generating set). With Saul Freedman, Andrea Lucchini and Colva Roney-Dougal, he found all groups for which the independence graph is the complement of the power graph (where two elements are joined if one is a power of the other).

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An example of the second type involves the independence graph (two elements joined if they are contained in a minimal generating set). With Saul Freedman, Andrea Lucchini and Colva Roney-Dougal, he found all groups for which the independence graph is the complement of the power graph (where two elements are joined if one is a power of the other). They also proved a similar but easier result on the relation between the rank graph (which Daniele described yesterday) and the enhanced power graph (coming up!)

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In the enhanced power graph, two vertices are joined if both are powers of the same element. Thus, the power graph is a spanning subgraph of the enhanced power graph.
The Gruenberg–Kegel graph of G has vertices the prime divisors of $|G|$, with an edge from p to q if G contains an element of order pq. (This is sometimes called the power graph, but it is not the power graph we met yesterday.)

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A group G is called an EPPO group if every element has prime power order. These were first investigated by Higman in the 1950s, who found the soluble EPPO groups; in the 1960s, Suzuki found the simple ones; and in 1981, Brandl found all these groups.

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Our proof, incidentally, resembles a classic alternating-paths argument from matching theory.

Super graphs on groups

In a paper with G. Arunkumar, Rajat Kanti Nath and Lavanya Selvaganesh, we proposed the following definition. If Γ is a type of graph defined on groups, then there is a super version of Γ, in which two elements x and y are joined if there exist conjugates x' and y' of x and y which are joined in Γ. (This is the conjugacy supergraph; a similar construction applies for other equivalence relations.)

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A Dedekind group is a group in which every subgroup is normal. Dedekind showed that such a finite group is either abelian, or of the form $Q \times A \times B$, where Q is the quaternion group of order 8, A an elementary abelian 2-group, and B an abelian group of odd order.

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Theorem

Let G be a finite group. Then the power graph and super power graph of G are equal if and only if G is a Dedekind group. The same holds for the enhanced power graph and the super enhanced power graph.

Super graphs, 2

A group G is a 2-Engel group if it satisfies the identity $[x, y, y] = 1$ for all $x, y \in G$, where $[x, y]$ is the commutator $x^{-1}y^{-1}xy$, and $[x, y, z] = [[x, y], z]$.

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A nilpotent group of class 2 satisfies the identity $[x,y,z] = 1$ for all $x, y, z \in G$, so is obviously 2-Engel. In the other direction, Hopkins and Levi independently showed that a 2-Engel group is nilpotent of class 3, and is "close" to being nilpotent of class 2.

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This uses two perhaps not well-known equivalents to the 2-Engel property: all centralizers are normal, and conjugate elements commute.

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We have $f(n) \geq \phi(n)$, where ϕ is Euler's function. This is not too much smaller than n (it is bounded below by $cn/\log\log n$), so the clique numbers of the two graphs are not too far apart.

A small detour

I can't resist mentioning a cute result here. Let $f(n)$ be the clique number of the power graph of a cyclic group of order n. This is an arithmetic function of n, and was calculated by Alireza, Ahmad and Abbas. But there is a nice estimate for it:

$$\phi(n) \leq f(n) \leq c\phi(n),$$

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where $c = 2.6481017597\ldots$; we might challenge number theorists to understand this number better!
Motivated by this, Sucharita Biswas, Angsuman Das, Hiranya Kishore Dey and I decided to look at what we called the difference graph and denoted by $D(G)$, on the grounds that it was expected to be fairly sparse, and we might possibly find graphs useful to network theorists.

Finding the jewel in the lotus

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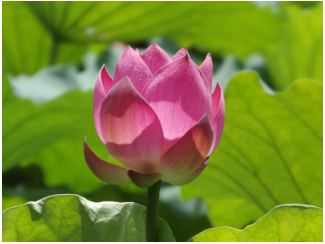

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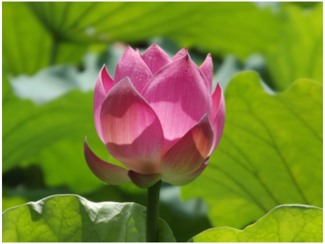

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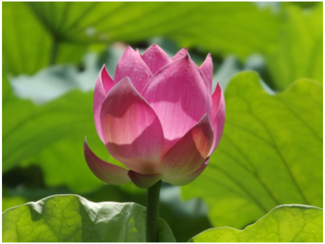

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Nearly ten years ago, Colva Roney-Dougal and I noticed that the automorphism group of the generating graph of A_5 (a group of order 60) has order 23482733690880. This impressively large group can almost all be stripped away.

Finding the jewel in the lotus, 2

Two vertices x and y of a graph Γ are twins if they have the same neighbours, apart possibly from one another. (Thus there are two kinds of twins; but this will not bother us.) Twin reduction is the process of repeatedly identifying twin vertices until no twins remain. It is not hard to show that the result of this process, up to isomorphism, does not depend on the order of the reductions. I will call this result the cokernel of Γ.

Recall that Γ is a cograph if it contains no induced 4-vertex path.

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Note that I have invented my own terminology here.

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We applied this process to the difference graph $D(G)$ of a group G, the graph whose edges are the edges of the enhanced power graph which are not in the power graph. We expect this to be a fairly sparse graph and potentially to contain interesting stuff.

But, as I said, I would expect this process to work for most types of graphs in groups.

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At this stage, we are doing "experimental mathematics". Empirically, simple groups G seem to fall into four types, as on the next slide.

Finding the jewel in the lotus, 4

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- Type 3: The cokernel of $D(G)$ consists of a large number of isomorphic small graphs, e.g. 253 or 325 copies of $K_5 - P_4$ in $\mathrm{PSL}(2,23)$ and $\mathrm{PSL}(2,25)$ respectively.

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But I am sure there is much more to be found. Please try your hand!

Three jewels

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- Let $G = \mathrm{PSL}(3,3)$. Then the cokernel of $D(G)$ is the following graph defined in the projective plane of order 3. The vertices are the ordered pairs (P, L) where P is a point and L a line. These are of two types: flags (where P and L are incident) and antiflags (where they are not). All edges join a flag to an antiflag; the antiflag (P, L) is incident with the flags (Q, M) where P is incident with M and Q with L. This graph has diameter 5 and girth 6, and 169 vertices.

- Let G be the Mathieu group M_{11}. Then the cokernel of $D(G)$ is a graph on 385 vertices; it is bipartite, with parts of size 165 and 220 (each of these sets an orbit of the automorphism group, which is just M_{11}), and has diameter 10 and girth 10. The valencies of vertices in the two parts are 4 and 3 respectively.

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… for your attention.