Designs on strongly regular graphs

Rosemary A. Bailey and Peter J. Cameron
University of St Andrews
with D. Ferreira, S. Ferreira and C. Nunes
UBI, Covilhã, Portugal

Ural Workshop on Group Theory and Combinatorics
26 October 2024

Our subject

Combinatorialists think of a design as a collection of subsets, but to a statistician it is more usually a partition (for example, a partition of the experimental units according to the treatment they will receive).

Our subject

Combinatorialists think of a design as a collection of subsets, but to a statistician it is more usually a partition (for example, a partition of the experimental units according to the treatment they will receive).
So we will be talking about partitions of the vertex set of a graph.

Our subject

Combinatorialists think of a design as a collection of subsets, but to a statistician it is more usually a partition (for example, a partition of the experimental units according to the treatment they will receive).
So we will be talking about partitions of the vertex set of a graph.
For the concept we are discussing,

Our subject

Combinatorialists think of a design as a collection of subsets, but to a statistician it is more usually a partition (for example, a partition of the experimental units according to the treatment they will receive).
So we will be talking about partitions of the vertex set of a graph.
For the concept we are discussing,

- some statisticians call it equivalent estimation;

Our subject

Combinatorialists think of a design as a collection of subsets, but to a statistician it is more usually a partition (for example, a partition of the experimental units according to the treatment they will receive).
So we will be talking about partitions of the vertex set of a graph.
For the concept we are discussing,

- some statisticians call it equivalent estimation;
- some other statisticians call it commutative orthogonal block structure;

Our subject

Combinatorialists think of a design as a collection of subsets, but to a statistician it is more usually a partition (for example, a partition of the experimental units according to the treatment they will receive).
So we will be talking about partitions of the vertex set of a graph.
For the concept we are discussing,

- some statisticians call it equivalent estimation;
- some other statisticians call it commutative orthogonal block structure;
- some combinatorialists call it an equitable partition of the graph.

Our subject

Combinatorialists think of a design as a collection of subsets, but to a statistician it is more usually a partition (for example, a partition of the experimental units according to the treatment they will receive).
So we will be talking about partitions of the vertex set of a graph.
For the concept we are discussing,

- some statisticians call it equivalent estimation;
- some other statisticians call it commutative orthogonal block structure;
- some combinatorialists call it an equitable partition of the graph.

Part of our message is that equitable partitions, a topic in combinatorics, is of practical use in Design of Experiments.

OBS and COBS

Four weeks ago, we told you about orthogonal block structures. Each of these is a sublattice of the partition lattice on a set Ω which consists of uniform partitions (all parts of the same size) and where the partitions, regarded as equivalence relations, commute.

OBS and COBS

Four weeks ago, we told you about orthogonal block structures. Each of these is a sublattice of the partition lattice on a set Ω which consists of uniform partitions (all parts of the same size) and where the partitions, regarded as equivalence relations, commute.

But this has very little to do with commutative orthogonal block structures. So this name is something of a misnomer for what we discuss here, although it is fairly widely used in some quarters.

Strongly regular graphs

Let Ω be a finite set of experimental units
(for example, plots in a field).

Strongly regular graphs

Let Ω be a finite set of experimental units
(for example, plots in a field).
In the experiments that RAB designs, there is always some kind of combinatorial structure on Ω.

Strongly regular graphs

Let Ω be a finite set of experimental units
(for example, plots in a field).
In the experiments that RAB designs, there is always some kind of combinatorial structure on Ω.
For this talk, the combinatorial structure is a graph Γ with vertex-set Ω.

Strongly regular graphs

Let Ω be a finite set of experimental units
(for example, plots in a field).
In the experiments that RAB designs, there is always some kind of combinatorial structure on Ω.
For this talk, the combinatorial structure is a graph Γ with vertex-set Ω.
This graph is regular if there is some constant d such that every vertex is contained in d edges.

Strongly regular graphs

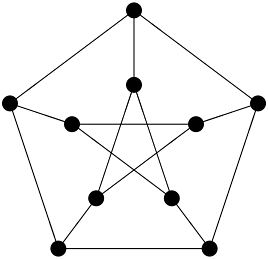
Let Ω be a finite set of experimental units
(for example, plots in a field).
In the experiments that RAB designs, there is always some kind of combinatorial structure on Ω.
For this talk, the combinatorial structure is a graph Γ with vertex-set Ω.
This graph is regular if there is some constant d such that every vertex is contained in d edges.

The graph Γ is strongly regular if

- it is regular;
- if two vertices are joined by an edge, then they have p common neighbours, for some constant p;
- if two vertices are not joined by an edge, then they have q common neighbours, for some constant q;
- the graph is neither complete nor null.

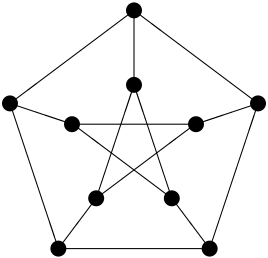
The Petersen graph

This is a famous strongly regular graph.



The Petersen graph

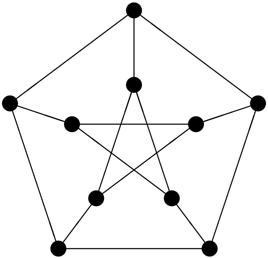
This is a famous strongly regular graph.



It has 10 vertices, each having degree 3.

The Petersen graph

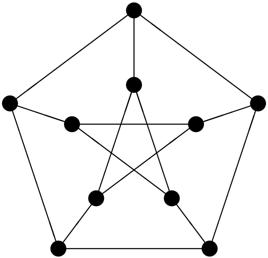
This is a famous strongly regular graph.



It has 10 vertices, each having degree 3.
If two vertices are joined by an edge, then they have no common neighbours.

The Petersen graph

This is a famous strongly regular graph.



It has 10 vertices, each having degree 3.
If two vertices are joined by an edge, then they have no common neighbours.
If two vertices are not joined by an edge, then they have exactly one common neighbour.

Commutative linear algebra

Suppose that Γ is a strongly regular graph with vertex-set Ω.

Commutative linear algebra

Suppose that Γ is a strongly regular graph with vertex-set Ω.
Here are three $\Omega \times \Omega$ real matrices associated with Γ:

- the adjacency matrix A has $A_{\alpha,\beta} = 1$ if $\{\alpha, \beta\}$ is an edge, and all other entries zero;
- the identity matrix I;
- the all-1 matrix J.

Commutative linear algebra

Suppose that Γ is a strongly regular graph with vertex-set Ω.
Here are three $\Omega \times \Omega$ real matrices associated with Γ:

- the adjacency matrix A has $A_{\alpha,\beta} = 1$ if $\{\alpha, \beta\}$ is an edge, and all other entries zero;
- the identity matrix I;
- the all-1 matrix J.

Because the graph Γ is strongly regular,
A^2 is a linear combination of A, I and J.

Commutative linear algebra

Suppose that Γ is a strongly regular graph with vertex-set Ω. Here are three $\Omega \times \Omega$ real matrices associated with Γ:

- the adjacency matrix A has $A_{\alpha,\beta} = 1$ if $\{\alpha, \beta\}$ is an edge, and all other entries zero;
- the identity matrix I;
- the all-1 matrix J.

Because the graph Γ is strongly regular,
A^2 is a linear combination of A, I and J.
In this case, the real vector space $\mathbb{R}^\Omega$ is the orthogonal direct sum of subspaces W_0, W_1 and W_2, each of which is (contained in) an eigenspace of A and an eigenspace of J, where W_0 is the one-dimensional subspace spanned by the all-1 vector $\mathbf{u}$.

Commutative linear algebra

Suppose that Γ is a strongly regular graph with vertex-set Ω.
Here are three $\Omega \times \Omega$ real matrices associated with Γ:

- the adjacency matrix A has $A_{\alpha,\beta} = 1$ if $\{\alpha, \beta\}$ is an edge, and all other entries zero;
- the identity matrix I;
- the all-1 matrix J.

Because the graph Γ is strongly regular,
A^2 is a linear combination of A, I and J.
In this case, the real vector space $\mathbb{R}^\Omega$ is the orthogonal direct sum of subspaces W_0, W_1 and W_2, each of which is (contained in) an eigenspace of A and an eigenspace of J, where W_0 is the one-dimensional subspace spanned by the all-1 vector $\mathbf{u}$.

Let G be the automorphism group of Γ. If Γ has rank 3 then G has precisely three orbits on $\Omega \times \Omega$. These correspond to the non-zero entries in the matrices I, A and $J - I - A$.

A walk around Design of Experiments

A combinatorial structure on a finite set $\rightarrow$

A walk around Design of Experiments

A combinatorial structure on a finite set $\rightarrow$
Commutative linear algebra over a finite-dimensional real vector space.

A walk around Design of Experiments

A combinatorial structure on a finite set $\to$
Commutative linear algebra over a finite-dimensional real vector space.

How should we design an experiment with certain numbers specified? $\to$

A walk around Design of Experiments

A combinatorial structure on a finite set $\rightarrow$
Commutative linear algebra over a finite-dimensional real vector space.

How should we design an experiment with certain numbers specified? $\rightarrow$
Assumptions about some relevant random variables $\rightarrow$

A walk around Design of Experiments

A combinatorial structure on a finite set $\rightarrow$
Commutative linear algebra over a finite-dimensional real vector space.

How should we design an experiment with certain numbers specified? $\rightarrow$
Assumptions about some relevant random variables $\rightarrow$
Eigenspaces, so back to linear algebra.

A walk around Design of Experiments

A combinatorial structure on a finite set $\rightarrow$
Commutative linear algebra over a finite-dimensional real vector space.

How should we design an experiment with certain numbers specified? $\rightarrow$
Assumptions about some relevant random variables $\rightarrow$
Eigenspaces, so back to linear algebra.

We will describe two different desirable statistical conditions that translate easily into combinatorics and linear algebra.

A walk around Design of Experiments

A combinatorial structure on a finite set $\rightarrow$
Commutative linear algebra over a finite-dimensional real vector space.

How should we design an experiment with certain numbers specified? $\rightarrow$
Assumptions about some relevant random variables $\rightarrow$
Eigenspaces, so back to linear algebra.

We will describe two different desirable statistical conditions that translate easily into combinatorics and linear algebra.

We will illustrate each of these conditions when applied to the same two combinatorial objects.

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f\colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f\colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

For each ω in Ω, there is a random variable Y_ω, which we will measure.

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f\colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

For each ω in Ω, there is a random variable Y_ω, which we will measure.

Assume that, for each treatment i, there is an unknown constant τ_i such that $\mathbb{E}(Y_\omega) = \tau_i$ if $f(\omega) = i$.

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f\colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

For each ω in Ω, there is a random variable Y_ω, which we will measure.

Assume that, for each treatment i, there is an unknown constant τ_i such that $\mathbb{E}(Y_\omega) = \tau_i$ if $f(\omega) = i$.

We randomize the design by applying a random permutation from G. When Γ has rank 3 this lets us assume that

$$\mathrm{Cov}(Y_\alpha, Y_\beta) = \begin{cases} \sigma^2 & \text{if } \alpha = \beta \\ \rho_1\sigma^2 & \text{if } \alpha \neq \beta \text{ and } \{\alpha, \beta\} \text{ is an edge of } \Gamma \\ \rho_2\sigma^2 & \text{otherwise.} \end{cases}$$

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f\colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

For each ω in Ω, there is a random variable Y_ω, which we will measure.

Assume that, for each treatment i, there is an unknown constant τ_i such that $\mathbb{E}(Y_\omega) = \tau_i$ if $f(\omega) = i$.

We randomize the design by applying a random permutation from G. When Γ has rank 3 this lets us assume that

$$\mathrm{Cov}(Y_\alpha, Y_\beta) = \begin{cases} \sigma^2 & \text{if } \alpha = \beta \\ \rho_1\sigma^2 & \text{if } \alpha \neq \beta \text{ and } \{\alpha, \beta\} \text{ is an edge of } \Gamma \\ \rho_2\sigma^2 & \text{otherwise.} \end{cases}$$

The eigenspaces of $\mathrm{Cov}(Y)$ are W_0, W_1 and W_2.

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f\colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

For each ω in Ω, there is a random variable Y_ω, which we will measure.

Assume that, for each treatment i, there is an unknown constant τ_i such that $\mathbb{E}(Y_\omega) = \tau_i$ if $f(\omega) = i$.

We randomize the design by applying a random permutation from G. When Γ has rank 3 this lets us assume that

$$\mathrm{Cov}(Y_\alpha, Y_\beta) = \begin{cases} \sigma^2 & \text{if } \alpha = \beta \\ \rho_1\sigma^2 & \text{if } \alpha \neq \beta \text{ and } \{\alpha, \beta\} \text{ is an edge of } \Gamma \\ \rho_2\sigma^2 & \text{otherwise.} \end{cases}$$

The eigenspaces of $\mathrm{Cov}(Y)$ are W_0, W_1 and W_2.

Call the corresponding eigenvalues γ_0, γ_1 and γ_2.

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f\colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

For each ω in Ω, there is a random variable Y_ω, which we will measure.

Assume that, for each treatment i, there is an unknown constant τ_i such that $\mathbb{E}(Y_\omega) = \tau_i$ if $f(\omega) = i$.

We randomize the design by applying a random permutation from G. When Γ has rank 3 this lets us assume that

$$\mathrm{Cov}(Y_\alpha, Y_\beta) = \begin{cases} \sigma^2 & \text{if } \alpha = \beta \\ \rho_1\sigma^2 & \text{if } \alpha \neq \beta \text{ and } \{\alpha, \beta\} \text{ is an edge of } \Gamma \\ \rho_2\sigma^2 & \text{otherwise.} \end{cases}$$

The eigenspaces of $\mathrm{Cov}(Y)$ are W_0, W_1 and W_2.

Call the corresponding eigenvalues γ_0, γ_1 and γ_2.

We do not know the values of γ_0, γ_1 and γ_2 in advance.

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f \colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

For each ω in Ω, there is a random variable Y_ω, which we will measure.

Assume that, for each treatment i, there is an unknown constant τ_i such that $\mathbb{E}(Y_\omega) = \tau_i$ if $f(\omega) = i$.

We randomize the design by applying a random permutation from G. When Γ has rank 3 this lets us assume that

$$\mathrm{Cov}(Y_\alpha, Y_\beta) = \begin{cases} \sigma^2 & \text{if } \alpha = \beta \\ \rho_1\sigma^2 & \text{if } \alpha \neq \beta \text{ and } \{\alpha, \beta\} \text{ is an edge of } \Gamma \\ \rho_2\sigma^2 & \text{otherwise.} \end{cases}$$

The eigenspaces of $\mathrm{Cov}(Y)$ are W_0, W_1 and W_2.

Call the corresponding eigenvalues γ_0, γ_1 and γ_2.

We do not know the values of γ_0, γ_1 and γ_2 in advance.

When is the choice of best design not affected by the values of γ_0, γ_1 and γ_2?

Two different desirable statistical conditions

Condition A We want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments.

Two different desirable statistical conditions

Condition A: We want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments.

Solution: Allocate the treatments to the vertices of Γ in such a way that, for all pairs $\{i, j\}$ of distinct treatments, there are λ edges with i at one end and j at the other.

Two different desirable statistical conditions

Condition A We want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments.

Solution Allocate the treatments to the vertices of Γ in such a way that, for all pairs $\{i, j\}$ of distinct treatments, there are λ edges with i at one end and j at the other.

Condition B We want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$. This is the difference between the averages for plots with treatment i and those with treatment j.

Two different desirable statistical conditions

Condition A We want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i,j\}$ of distinct treatments.

Solution Allocate the treatments to the vertices of Γ in such a way that, for all pairs $\{i,j\}$ of distinct treatments, there are λ edges with i at one end and j at the other.

Condition B We want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$. This is the difference between the averages for plots with treatment i and those with treatment j.

Solution The subspace V_T of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each treatment can be orthogonally decomposed as

$$W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2).$$

Two conditions, two types of structure

We are going to discuss the Conditions A and B which we have just defined in two families of strongly regular graphs with rank 3:

Two conditions, two types of structure

We are going to discuss the Conditions A and B which we have just defined in two families of strongly regular graphs with rank 3:

- Combinatorial structure 1: Disjoint union of b complete graphs each of size k.

Two conditions, two types of structure

We are going to discuss the Conditions A and B which we have just defined in two families of strongly regular graphs with rank 3:

- Combinatorial structure 1: Disjoint union of b complete graphs each of size k.
- Combinatorial structure 2: Triangular graph $T(m)$, the line graph of the complete graph K_m.

Two conditions, two types of structure

We are going to discuss the Conditions A and B which we have just defined in two families of strongly regular graphs with rank 3:

- Combinatorial structure 1: Disjoint union of b complete graphs each of size k.
- Combinatorial structure 2: Triangular graph $T(m)$, the line graph of the complete graph K_m.

We are mainly interested in Condition B, but in each case we describe designs satisfying Condition A for comparison.

Two conditions, two types of structure

We are going to discuss the Conditions A and B which we have just defined in two families of strongly regular graphs with rank 3:

- Combinatorial structure 1: Disjoint union of b complete graphs each of size k.
- Combinatorial structure 2: Triangular graph $T(m)$, the line graph of the complete graph K_m.

We are mainly interested in Condition B, but in each case we describe designs satisfying Condition A for comparison. Condition B is the one referred to as Commutative Orthogonal Block Structure or COBS by some statisticians.

Combinatorial Structure 1: Partition into Blocks

This is probably the best-known combinatorial structure in Design of Experiments.
The set Ω is partitioned into b blocks, each of size k.
Thus Γ consists of b disjoint copies of the complete graph on k vertices. This is strongly regular.

Combinatorial Structure 1: Partition into Blocks

This is probably the best-known combinatorial structure in Design of Experiments.

The set Ω is partitioned into b blocks, each of size k.

Thus Γ consists of b disjoint copies of the complete graph on k vertices. This is strongly regular.

For Condition A, we want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments.

Combinatorial Structure 1: Partition into Blocks

This is probably the best-known combinatorial structure in Design of Experiments.
The set Ω is partitioned into b blocks, each of size k.
Thus Γ consists of b disjoint copies of the complete graph on k vertices. This is strongly regular.
For Condition A, we want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments. If $\gamma_1 \neq \gamma_2$ and $k < t$ then the only way to achieve this is to use a balanced incomplete-block design.

Combinatorial Structure 1: Partition into Blocks

This is probably the best-known combinatorial structure in Design of Experiments.
The set Ω is partitioned into b blocks, each of size k.
Thus Γ consists of b disjoint copies of the complete graph on k vertices. This is strongly regular.
For Condition A, we want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments. If $\gamma_1 \neq \gamma_2$ and $k < t$ then the only way to achieve this is to use a balanced incomplete-block design.
This means that each treatment occurs no more than once in each block, and there is an integer λ such that, for all pairs $\{i, j\}$ of distinct treatments, there are λ blocks in which i and j both occur.

Combinatorial Structure 1: Partition into Blocks

This is probably the best-known combinatorial structure in Design of Experiments.
The set Ω is partitioned into b blocks, each of size k.
Thus Γ consists of b disjoint copies of the complete graph on k vertices. This is strongly regular.
For Condition A, we want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments. If $\gamma_1 \neq \gamma_2$ and $k < t$ then the only way to achieve this is to use a balanced incomplete-block design.
This means that each treatment occurs no more than once in each block, and there is an integer λ such that, for all pairs $\{i, j\}$ of distinct treatments, there are λ blocks in which i and j both occur.
If $k = t$ then each block must contain every treatment.

Combinatorial Structure 1: Partition into Blocks

This is probably the best-known combinatorial structure in Design of Experiments.

The set Ω is partitioned into b blocks, each of size k.

Thus Γ consists of b disjoint copies of the complete graph on k vertices. This is strongly regular.

For Condition A, we want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments. If $\gamma_1 \neq \gamma_2$ and $k < t$ then the only way to achieve this is to use a balanced incomplete-block design.

This means that each treatment occurs no more than once in each block, and there is an integer λ such that, for all pairs $\{i, j\}$ of distinct treatments, there are λ blocks in which i and j both occur.

If $k = t$ then each block must contain every treatment.

If $k > t$ then something slightly more complicated is needed.

An example of a balanced incomplete-block design

Here is a balanced incomplete-block design with $b = 14$, $k = 4$, $t = 8$ and $\lambda = 3$.

1	3	5	7	2	4	6	8
1	2	5	6	3	4	7	8
1	2	3	4	5	6	7	8
1	4	5	8	2	3	6	7
1	3	6	8	2	4	5	7
1	2	7	8	3	4	5	6
1	4	6	7	2	3	5	8

Partition into Blocks: Condition B

The set Ω is partitioned into b blocks, each of size k.

Partition into Blocks: Condition B

The set Ω is partitioned into b blocks, each of size k.
Let V_B be the b-dimensional subspace of $\mathbb{R}^{\Omega}$ consisting of vectors which are constant on each block.

Partition into Blocks: Condition B

The set Ω is partitioned into b blocks, each of size k.
Let V_B be the b-dimensional subspace of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each block.
Then $W_0 = \langle \mathbf{u} \rangle$, $W_1 = V_B \cap W_0^\perp$ and $W_2 = V_B^\perp$.

Partition into Blocks: Condition B

The set Ω is partitioned into b blocks, each of size k.
Let V_B be the b-dimensional subspace of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each block.
Then $W_0 = \langle \mathbf{u} \rangle$, $W_1 = V_B \cap W_0^\perp$ and $W_2 = V_B^\perp$.

Condition B We want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$.
This is the difference between the averages for plots with treatment i and those with treatment j.

Partition into Blocks: Condition B

The set Ω is partitioned into b blocks, each of size k.
Let V_B be the b-dimensional subspace of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each block.
Then $W_0 = \langle \mathbf{u} \rangle$, $W_1 = V_B \cap W_0^\perp$ and $W_2 = V_B^\perp$.

Condition B We want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$. This is the difference between the averages for plots with treatment i and those with treatment j.

Since the treatment subspace V_T contains W_0, there are three possibilities.

Partition into Blocks: Condition B

The set Ω is partitioned into b blocks, each of size k.
Let V_B be the b-dimensional subspace of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each block.
Then $W_0 = \langle \mathbf{u} \rangle$, $W_1 = V_B \cap W_0^\perp$ and $W_2 = V_B^\perp$.

Condition B We want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$. This is the difference between the averages for plots with treatment i and those with treatment j.

Since the treatment subspace V_T contains W_0, there are three possibilities.

(a) $V_T \leq W_0 \oplus W_2$.

(b) $V_T \leq W_0 \oplus W_1$.

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.
There are k treatments, and each occurs exactly once in each block. This is called a complete-block design.

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.
There are k treatments, and each occurs exactly once in each block. This is called a complete-block design.
For example, when $b = 4$ and $k = 3$ we get

1	2	3	1	2	3	1	2	3	1	2	3

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.
There are k treatments, and each occurs exactly once in each block. This is called a complete-block design.
For example, when $b = 4$ and $k = 3$ we get

1	2	3	1	2	3	1	2	3	1	2	3

More generally, any subset of treatments may be merged into a single treatment. For example,

1	2	2	1	2	2	1	2	2	1	2	2

Solution (b) for Condition B

(b) $V_T \leq W_0 \oplus W_1$.

Solution (b) for Condition B

(b) $V_T \leq W_0 \oplus W_1$.
There are t treatments, where t divides b. Each treatment is applied to every plot in each of b/t whole blocks.

Solution (b) for Condition B

(b) $V_T \leq W_0 \oplus W_1$.
There are t treatments, where t divides b. Each treatment is applied to every plot in each of b/t whole blocks.
For example, when $b = 4$, $k = 3$ and $t = 2$ we get

A	A	A	B	B	B	A	A	A	B	B	B

Solution (b) for Condition B

(b) $V_T \leq W_0 \oplus W_1$.
There are t treatments, where t divides b. Each treatment is applied to every plot in each of b/t whole blocks.
For example, when $b = 4$, $k = 3$ and $t = 2$ we get

A	A	A	B	B	B	A	A	A	B	B	B

Such designs are used when management constraints make it impractical to apply the treatments to the individual plots.

Solution (c) for Condition B

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.

Solution (c) for Condition B

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
We combine the two previous approaches.
The treatment set is $\mathcal{T}_1 \times \mathcal{T}_2$,
where $|\mathcal{T}_1| = t_1$, which divides b, and $|\mathcal{T}_2| = k$.

Solution (c) for Condition B

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
We combine the two previous approaches.
The treatment set is $\mathcal{T}_1 \times \mathcal{T}_2$,
where $|\mathcal{T}_1| = t_1$, which divides b, and $|\mathcal{T}_2| = k$.
Each item from $\mathcal{T}_2$ is applied to one plot per block.

Solution (c) for Condition B

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
We combine the two previous approaches.
The treatment set is $\mathcal{T}_1 \times \mathcal{T}_2$,
where $|\mathcal{T}_1| = t_1$, which divides b, and $|\mathcal{T}_2| = k$.
Each item from $\mathcal{T}_2$ is applied to one plot per block.
Each item from $\mathcal{T}_1$ is applied to b/t_1 whole blocks.

Solution (c) for Condition B

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
We combine the two previous approaches.
The treatment set is $\mathcal{T}_1 \times \mathcal{T}_2$,
where $|\mathcal{T}_1| = t_1$, which divides b, and $|\mathcal{T}_2| = k$.
Each item from $\mathcal{T}_2$ is applied to one plot per block.
Each item from $\mathcal{T}_1$ is applied to b/t_1 whole blocks.
For example, when $b = 4$, $k = 3$, $t = 6$ and $t_1 = 2$ we get

$A1$	$A2$	$A3$

$B1$	$B2$	$B3$

$A1$	$A2$	$A3$

$B1$	$B2$	$B3$

Solution (c) for Condition B

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
We combine the two previous approaches.
The treatment set is $\mathcal{T}_1 \times \mathcal{T}_2$,
where $|\mathcal{T}_1| = t_1$, which divides b, and $|\mathcal{T}_2| = k$.
Each item from $\mathcal{T}_2$ is applied to one plot per block.
Each item from $\mathcal{T}_1$ is applied to b/t_1 whole blocks.
For example, when $b = 4$, $k = 3$, $t = 6$ and $t_1 = 2$ we get

$A1$	$A2$	$A3$	$B1$	$B2$	$B3$	$A1$	$A2$	$A3$	$B1$	$B2$	$B3$

These are called split-plot designs, and are widely used in practice.

Combinatorial Structure 2: Half-Diallel

Diallel experiments are traditionally used in plant breeding. In a full diallel experiment, the experimental units are all ordered crosses between m parental lines.

Combinatorial Structure 2: Half-Diallel

Diallel experiments are traditionally used in plant breeding. In a full diallel experiment, the experimental units are all ordered crosses between m parental lines.

In situtations where the gender of the parent is irrelevant, it is efficient to use half-diallel experiments, in which the experimental units consist of all unordered crosses between m parental lines, excluding self-crosses.

Combinatorial Structure 2: Half-Diallel

Diallel experiments are traditionally used in plant breeding. In a full diallel experiment, the experimental units are all ordered crosses between m parental lines.

In situtations where the gender of the parent is irrelevant, it is efficient to use half-diallel experiments, in which the experimental units consist of all unordered crosses between m parental lines, excluding self-crosses.

This structure is also useful in experiments where pairs of individuals are required to complete some task, with both individuals playing the same role.

Combinatorial Structure 2: Half-Diallel

Diallel experiments are traditionally used in plant breeding. In a full diallel experiment, the experimental units are all ordered crosses between m parental lines.
In situtations where the gender of the parent is irrelevant, it is efficient to use half-diallel experiments, in which the experimental units consist of all unordered crosses between m parental lines, excluding self-crosses.

This structure is also useful in experiments where pairs of individuals are required to complete some task, with both individuals playing the same role.
This happens in some experiments in human-computer interaction.

Combinatorial Structure 2: Half-Diallel

Diallel experiments are traditionally used in plant breeding. In a full diallel experiment, the experimental units are all ordered crosses between m parental lines.
In situtations where the gender of the parent is irrelevant, it is efficient to use half-diallel experiments, in which the experimental units consist of all unordered crosses between m parental lines, excluding self-crosses.

This structure is also useful in experiments where pairs of individuals are required to complete some task, with both individuals playing the same role.
This happens in some experiments in human-computer interaction.
For example, the aim of the experiment might be to compare different methods for researchers to collaborate when they are unable to meet face-to-face, such as email, online meetings, old-fashioned letters, telephone calls with and without video.

Combinatorial Structure 2: more detail

Now the set Ω consists of all unordered pairs from the set $\{1, 2, \ldots, m\}$ of m distinct individuals, where $m \geq 4$.

Combinatorial Structure 2: more detail

Now the set Ω consists of all unordered pairs from the set $\{1, 2, \ldots, m\}$ of m distinct individuals, where $m \geq 4$.
These form the vertices of the graph Γ.

Combinatorial Structure 2: more detail

Now the set Ω consists of all unordered pairs from the set $\{1, 2, \ldots, m\}$ of m distinct individuals, where $m \geq 4$.
These form the vertices of the graph Γ.
There is an edge between two distinct vertices if and only if they have an individual in common.

Combinatorial Structure 2: more detail

Now the set Ω consists of all unordered pairs from the set $\{1, 2, \ldots, m\}$ of m distinct individuals, where $m \geq 4$.
These form the vertices of the graph Γ.
There is an edge between two distinct vertices if and only if they have an individual in common.
Thus every vertex is joined to $2(m-2)$ other vertices.

Combinatorial Structure 2: more detail

Now the set Ω consists of all unordered pairs from the set $\{1, 2, \ldots, m\}$ of m distinct individuals, where $m \geq 4$.
These form the vertices of the graph Γ.
There is an edge between two distinct vertices if and only if they have an individual in common.
Thus every vertex is joined to $2(m-2)$ other vertices.
This is called the triangular graph $T(m)$.

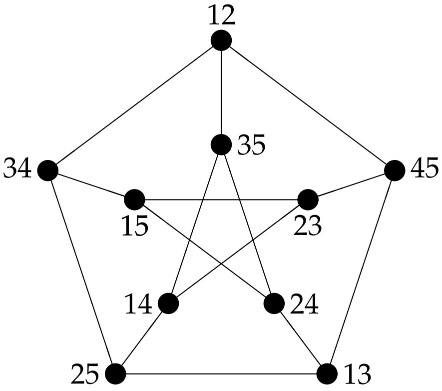
Combinatorial Structure 2: more detail

Now the set Ω consists of all unordered pairs from the set $\{1, 2, \ldots, m\}$ of m distinct individuals, where $m \geq 4$.
These form the vertices of the graph Γ.
There is an edge between two distinct vertices if and only if they have an individual in common.
Thus every vertex is joined to $2(m-2)$ other vertices.
This is called the triangular graph $T(m)$.
It is strongly regular, and its adjacency matrix A satisfies

$$A^2 = (2m-8)I + (m-6)A + 4J.$$

The Petersen graph again

This labelling of the vertices shows that it is the complement of the triangular graph $T(5)$.



How to picture the vertices of $T(m)$ in general

When $m = 6$ the set Ω has 15 elements, which can be shown as the cells of a 6×6 square lying below the main diagonal.

How to picture the vertices of $T(m)$ in general

When $m = 6$ the set Ω has 15 elements, which can be shown as the cells of a 6×6 square lying below the main diagonal.

	1	2	3	4	5
2					
3					
4					
5					
6					

How to picture the vertices of $T(m)$ in general

When $m = 6$ the set Ω has 15 elements, which can be shown as the cells of a 6×6 square lying below the main diagonal.

	1	2	3	4	5
2					
3					
4					
5			$*$		
6					

$$* = \{3, 5\}$$

How to picture the vertices of $T(m)$ in general

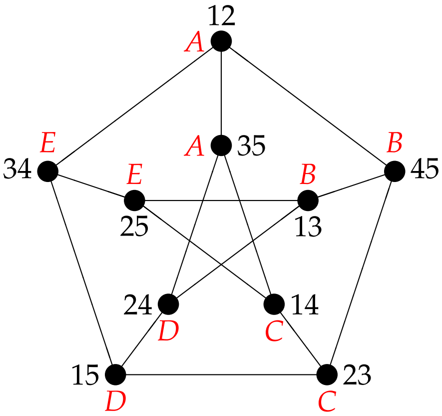
When $m = 6$ the set Ω has 15 elements, which can be shown as the cells of a 6×6 square lying below the main diagonal.

	1	2	3	4	5
2					
3	$\circ$	$\circ$			
4			$\circ$		
5	$\circ$	$\circ$	$*$	$\circ$	
6			$\circ$		$\circ$

$$* = \{3, 5\}$$

$$\circ = \text{vertices joined to vertex } \{3, 5\}$$

Condition A on the Petersen graph



For each treatment, there is one edge that has that treatment on both vertices.
For each pair of distinct treatments, there is one edge that has them on its endpoints.

Triangular Graph: Condition B

The set Ω consists of all unordered pairs from $\{1, 2, \ldots, m\}$.

Triangular Graph: Condition B

The set Ω consists of all unordered pairs from $\{1, 2, \ldots, m\}$. For $i = 1, \ldots, m$, let $\mathbf{v}_i$ be the vector taking the value 1 on each pair that includes individual i and value 0 elsewhere. Let V_{ind} be the m-dimensional subspace of $\mathbb{R}^\Omega$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

Triangular Graph: Condition B

The set Ω consists of all unordered pairs from $\{1, 2, \ldots, m\}$. For $i = 1, \ldots, m$, let $\mathbf{v}_i$ be the vector taking the value 1 on each pair that includes individual i and value 0 elsewhere. Let V_{ind} be the m-dimensional subspace of $\mathbb{R}^\Omega$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Then $W_0 = \langle \mathbf{u} \rangle$, $W_1 = V_{\text{ind}} \cap W_0^\perp$ and $W_2 = V_{\text{ind}}^\perp$.

Triangular Graph: Condition B

The set Ω consists of all unordered pairs from $\{1, 2, \ldots, m\}$. For $i = 1, \ldots, m$, let $\mathbf{v}_i$ be the vector taking the value 1 on each pair that includes individual i and value 0 elsewhere. Let V_{ind} be the m-dimensional subspace of $\mathbb{R}^\Omega$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Then $W_0 = \langle \mathbf{u} \rangle$, $W_1 = V_{\text{ind}} \cap W_0^\perp$ and $W_2 = V_{\text{ind}}^\perp$.

For Condition B, we want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$.

Triangular Graph: Condition B

The set Ω consists of all unordered pairs from $\{1, 2, \ldots, m\}$. For $i = 1, \ldots, m$, let $\mathbf{v}_i$ be the vector taking the value 1 on each pair that includes individual i and value 0 elsewhere. Let V_{ind} be the m-dimensional subspace of $\mathbb{R}^\Omega$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$.
Then $W_0 = \langle \mathbf{u} \rangle$, $W_1 = V_{\text{ind}} \cap W_0^\perp$ and $W_2 = V_{\text{ind}}^\perp$.
For Condition B, we want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$.
This is the difference between the averages for plots with treatment i and those with treatment j.

Triangular Graph: Condition B

The set Ω consists of all unordered pairs from $\{1, 2, \ldots, m\}$. For $i = 1, \ldots, m$, let $\mathbf{v}_i$ be the vector taking the value 1 on each pair that includes individual i and value 0 elsewhere. Let V_{ind} be the m-dimensional subspace of $\mathbb{R}^\Omega$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

Then $W_0 = \langle \mathbf{u} \rangle$, $W_1 = V_{\text{ind}} \cap W_0^\perp$ and $W_2 = V_{\text{ind}}^\perp$.

For Condition B, we want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$.

This is the difference between the averages for plots with treatment i and those with treatment j.

Since V_T contains W_0, there are three possibilities.

Triangular Graph: Condition B

The set Ω consists of all unordered pairs from $\{1, 2, \ldots, m\}$. For $i = 1, \ldots, m$, let $\mathbf{v}_i$ be the vector taking the value 1 on each pair that includes individual i and value 0 elsewhere. Let V_{ind} be the m-dimensional subspace of $\mathbb{R}^\Omega$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$.
Then $W_0 = \langle \mathbf{u} \rangle$, $W_1 = V_{\text{ind}} \cap W_0^\perp$ and $W_2 = V_{\text{ind}}^\perp$.
For Condition B, we want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$.
This is the difference between the averages for plots with treatment i and those with treatment j.
Since V_T contains W_0, there are three possibilities.

(a) $V_T \leq W_0 \oplus W_2$.

(b) $V_T \leq W_0 \oplus W_1$.

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.
For treatment A, let p_{Ai} be the number of pairs including individual i on which A occurs. We were able to show that if (a) holds then

- $p_{Ai} = p_{Aj} = p_A$ for all individuals i and j;

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.
For treatment A, let p_{Ai} be the number of pairs including individual i on which A occurs. We were able to show that if (a) holds then

- $p_{Ai} = p_{Aj} = p_A$ for all individuals i and j;
- treatment A occurs on $mp_A/2$ pairs, and so mp_A is even for all treatments A;

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.
For treatment A, let p_{Ai} be the number of pairs including individual i on which A occurs. We were able to show that if (a) holds then

- $p_{Ai} = p_{Aj} = p_A$ for all individuals i and j;
- treatment A occurs on $mp_A/2$ pairs, and so mp_A is even for all treatments A;
- if $p_A = 1$ then m is even and A occurs on $m/2$ pairs;

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.
For treatment A, let p_{Ai} be the number of pairs including individual i on which A occurs. We were able to show that if (a) holds then

- $p_{Ai} = p_{Aj} = p_A$ for all individuals i and j;
- treatment A occurs on $mp_A/2$ pairs, and so mp_A is even for all treatments A;
- if $p_A = 1$ then m is even and A occurs on $m/2$ pairs;
- if this is true for all treatments then $t = m - 1$.

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.
For treatment A, let p_{Ai} be the number of pairs including individual i on which A occurs. We were able to show that if (a) holds then
- $p_{Ai} = p_{Aj} = p_A$ for all individuals i and j;
- treatment A occurs on $mp_A/2$ pairs, and so mp_A is even for all treatments A;
- if $p_A = 1$ then m is even and A occurs on $m/2$ pairs;
- if this is true for all treatments then $t = m - 1$.

In this case, we can do this by using a symmetric Latin square of order m with a single letter on the main diagonal and omitting the main diagonal and plots above the main diagonal.

Solution (a) for Condition B

(a) $V_T \leq W_0 \oplus W_2$.
For treatment A, let p_{Ai} be the number of pairs including individual i on which A occurs. We were able to show that if (a) holds then

- $p_{Ai} = p_{Aj} = p_A$ for all individuals i and j;
- treatment A occurs on $mp_A/2$ pairs, and so mp_A is even for all treatments A;
- if $p_A = 1$ then m is even and A occurs on $m/2$ pairs;
- if this is true for all treatments then $t = m - 1$.

In this case, we can do this by using a symmetric Latin square of order m with a single letter on the main diagonal and omitting the main diagonal and plots above the main diagonal.
(Start with a symmetric idempotent Latin square; add an extra row at the bottom; move every diagonal element down to the bottom row; then put a dummy like ∞ on every diagonal cell.)

An example with $m = 8$

	1	2	3	4	5	6	7
2	C						
3	D	E					
4	E	F	G				
5	F	G	A	B			
6	G	A	B	C	D		
7	A	B	C	D	E	F	
8	B	D	F	A	C	E	G

An example with $m = 8$

	1	2	3	4	5	6	7
2	C						
3	D	E					
4	E	F	G				
5	F	G	A	B			
6	G	A	B	C	D		
7	A	B	C	D	E	F	
8	B	D	F	A	C	E	G

Each treatment occurs exactly once with each individual.

An example with $m = 8$

	1	2	3	4	5	6	7
2	C						
3	D	E					
4	E	F	G				
5	F	G	A	B			
6	G	A	B	C	D		
7	A	B	C	D	E	F	
8	B	D	F	A	C	E	G

Each treatment occurs exactly once with each individual.

Just as with complete-block designs, any subset of treatments may be merged into a single treatment.

Solution (a) for Condition B when m is odd

When m is odd, p_A must even for every treatment A.

Solution (a) for Condition B when m is odd

When m is odd, p_A must even for every treatment A.
If $p_A = 2$ for every treatment A then $m = 2t + 1$.

Solution (a) for Condition B when m is odd

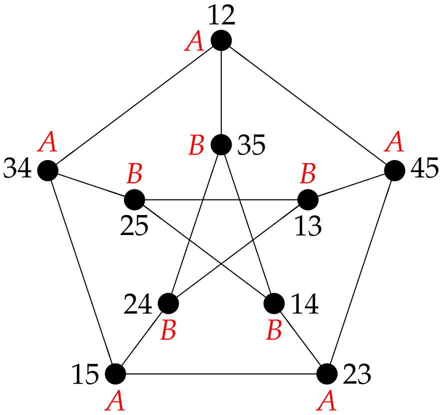
When m is odd, p_A must even for every treatment A.
If $p_A = 2$ for every treatment A then $m = 2t + 1$.
Now label the treatments by $\{1, 2, \ldots, t\}$.
The treatment applied to the pair $\{i, j\}$ is whichever is smaller of the differences $i - j$ and $j - i$ modulo m.

Solution (a) for Condition B when m is odd

When m is odd, p_A must even for every treatment A.
If $p_A = 2$ for every treatment A then $m = 2t + 1$.
Now label the treatments by $\{1, 2, \ldots, t\}$.
The treatment applied to the pair $\{i, j\}$ is whichever is smaller of the differences $i - j$ and $j - i$ modulo m.
When $m = 9$ this gives

	1	2	3	4	5	6	7	8
2	1							
3	2	1						
4	3	2	1					
5	4	3	2	1				
6	4	4	3	2	1			
7	3	4	4	3	2	1		
8	2	3	4	4	3	2	1	
9	1	2	3	4	4	3	2	1

Solution (a) for Condition B when $m = 5$



Here A represents ± 1 mod 5 and B represents ± 2 mod 5.

Solution (b) for Condition B

(b) $V_T \leq W_0 \oplus W_1$.

Solution (b) for Condition B

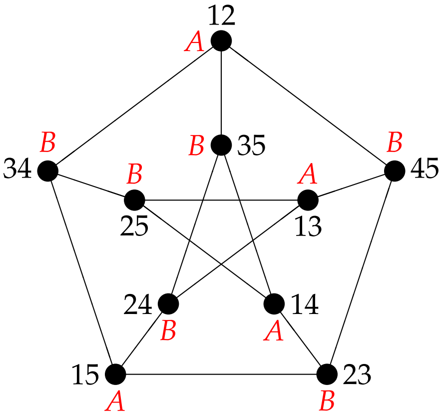
(b) $V_T \leq W_0 \oplus W_1$.
There is essentially only one solution.
There are precisely two treatments, say A and B. There is one special individual i. Treatment A is applied to all pairs containing i, and treatment B is applied to all other pairs.

Solution (b) for Condition B

(b) $V_T \leq W_0 \oplus W_1$.
There is essentially only one solution.
There are precisely two treatments, say A and B. There is one special individual i. Treatment A is applied to all pairs containing i, and treatment B is applied to all other pairs.
When $m = 9$ this gives

	1	2	3	4	5	6	7	8
2	*A*							
3	*A*	*B*						
4	*A*	*B*	*B*					
5	*A*	*B*	*B*	*B*				
6	*A*	*B*	*B*	*B*	*B*			
7	*A*	*B*	*B*	*B*	*B*	*B*		
8	*A*	*B*	*B*	*B*	*B*	*B*	*B*	
9	*A*	*B*	*B*	*B*	*B*	*B*	*B*	*B*

Solution (b) for Condition B when $m = 5$



The two treatments are not equally replicated.

Solution (c) for Condition 2

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.

Solution (c) for Condition 2

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
Here is a very general solution.

- Partition the set of individuals into n sorts $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of size $s_1, \ldots, s_n$, where $n \geq 2$.

Solution (c) for Condition 2

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
Here is a very general solution.

- Partition the set of individuals into n sorts $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of size $s_1, \ldots, s_n$, where $n \geq 2$.
- If $s_i > 1$ then put a solution (a) design on pairs of individuals of sort i, using t_i treatments forming a set $\mathcal{T}_i$.

Solution (c) for Condition 2

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
Here is a very general solution.

- Partition the set of individuals into n sorts $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of size $s_1, \ldots, s_n$, where $n \geq 2$.
- If $s_i > 1$ then put a solution (a) design on pairs of individuals of sort i, using t_i treatments forming a set $\mathcal{T}_i$.
- If $s_i = 2$ then $\mathcal{T}_i$ has a single treatment with replication 1, so avoid this case.

Solution (c) for Condition 2

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
Here is a very general solution.

- Partition the set of individuals into n sorts $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of size $s_1, \ldots, s_n$, where $n \geq 2$.
- If $s_i > 1$ then put a solution (a) design on pairs of individuals of sort i, using t_i treatments forming a set $\mathcal{T}_i$.
- If $s_i = 2$ then $\mathcal{T}_i$ has a single treatment with replication 1, so avoid this case.
- If $s_i = 3$ then the only way to avoid replication 1 is to have $t_i = 1$.

Solution (c) for Condition 2

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
Here is a very general solution.

- Partition the set of individuals into n sorts $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of size $s_1, \ldots, s_n$, where $n \geq 2$.
- If $s_i > 1$ then put a solution (a) design on pairs of individuals of sort i, using t_i treatments forming a set $\mathcal{T}_i$.
- If $s_i = 2$ then $\mathcal{T}_i$ has a single treatment with replication 1, so avoid this case.
- If $s_i = 3$ then the only way to avoid replication 1 is to have $t_i = 1$.
- If $n = 2$ and $s_1 = 1$ then make sure that $t_2 > 1$, to avoid solution (b).

Solution (c) for Condition 2

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
$V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
Here is a very general solution.

- Partition the set of individuals into n sorts $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of size $s_1, \ldots, s_n$, where $n \geq 2$.
- If $s_i > 1$ then put a solution (a) design on pairs of individuals of sort i, using t_i treatments forming a set $\mathcal{T}_i$.
- If $s_i = 2$ then $\mathcal{T}_i$ has a single treatment with replication 1, so avoid this case.
- If $s_i = 3$ then the only way to avoid replication 1 is to have $t_i = 1$.
- If $n = 2$ and $s_1 = 1$ then make sure that $t_2 > 1$, to avoid solution (b).
- If $i < j$ then let t_{ij} be any common divisor of s_i and s_j. Make a set $\mathcal{T}_{ij}$ of t_{ij} treatments. Allocate these to the cells in the rectangle $\mathcal{S}_j \times \mathcal{S}_i$ in such a way that all treatments appear equally often in each row and equally often in each column.

Solution (c) for Condition 2

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
Here is a very general solution.

- Partition the set of individuals into n sorts $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of size $s_1, \ldots, s_n$, where $n \geq 2$.
- If $s_i > 1$ then put a solution (a) design on pairs of individuals of sort i, using t_i treatments forming a set $\mathcal{T}_i$.
- If $s_i = 2$ then $\mathcal{T}_i$ has a single treatment with replication 1, so avoid this case.
- If $s_i = 3$ then the only way to avoid replication 1 is to have $t_i = 1$.
- If $n = 2$ and $s_1 = 1$ then make sure that $t_2 > 1$, to avoid solution (b).
- If $i < j$ then let t_{ij} be any common divisor of s_i and s_j. Make a set $\mathcal{T}_{ij}$ of t_{ij} treatments. Allocate these to the cells in the rectangle $\mathcal{S}_j \times \mathcal{S}_i$ in such a way that all treatments appear equally often in each row and equally often in each column.
- If $i < j$ and $s_i = s_j = 1$ then $\mathcal{T}_{ij}$ has a single treatment with replication 1, so avoid this case.

Theorem about this solution

Theorem

For $i = 1, \ldots, n$,
let $\mathbf{w}_i$ *be the vector whose entries are*

$$\begin{cases} 0 & \textit{on all pairs which do not involve an individual of sort } i \\ 1 & \textit{on all pairs which involve a single individual of sort } i \\ 2 & \textit{on all pairs which involve two indiviudals of sort } i \end{cases}$$

Theorem about this solution

Theorem

For $i = 1, \ldots, n$,
let $\mathbf{w}_i$ be the vector whose entries are

$$\begin{cases} 0 & \text{on all pairs which do not involve an individual of sort } i \\ 1 & \text{on all pairs which involve a single individual of sort } i \\ 2 & \text{on all pairs which involve two indiviudals of sort } i \end{cases}$$

Then

- *The vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ span an n-dimensional subspace of $V_T \cap (W_0 \oplus W_1)$.*

Theorem about this solution

Theorem

For $i = 1, \ldots, n$,
let $\mathbf{w}_i$ *be the vector whose entries are*

$$\begin{cases} 0 & \textit{on all pairs which do not involve an individual of sort } i \\ 1 & \textit{on all pairs which involve a single individual of sort } i \\ 2 & \textit{on all pairs which involve two indiviudals of sort } i \end{cases}$$

Then

- *The vectors* $\mathbf{w}_1, \ldots, \mathbf{w}_n$ *span an* n*-dimensional subspace of* $V_T \cap (W_0 \oplus W_1)$.
- *If* $\mathbf{v} \in V_T$ *is orthogonal to* $\mathbf{w}_i$ *for* $i = 1, \ldots, n$ *then* $\mathbf{v} \in W_2$.

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1, 2, 3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1,2,3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.
$\mathcal{S}_2 = \{4,5,6,7,8,9\}$, $\mathcal{T}_2 = \{E,F,G,H,I\}$ and $t_2 = 5$.

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1,2,3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.
$\mathcal{S}_2 = \{4,5,6,7,8,9\}$, $\mathcal{T}_2 = \{E,F,G,H,I\}$ and $t_2 = 5$.
$\mathcal{T}_{12} = \{B,C,D\}$ and $t_{12} = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B,C,D\}$ and $t_2 = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B, C, D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J, K, L\}$ and $t_3 = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B, C, D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J, K, L\}$ and $t_3 = 3$.
$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B,C,D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J,K,L\}$ and $t_3 = 3$.
$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$. $\mathcal{T}_{13} = \{E\}$ and $t_{13} = 1$.

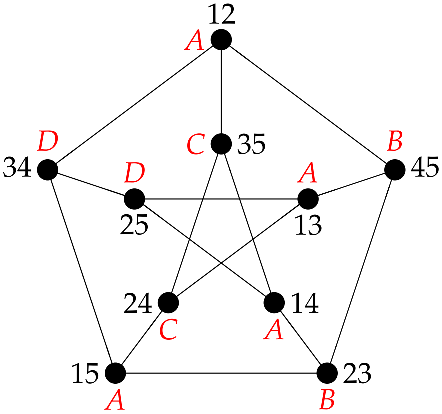
An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B,C,D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J,K,L\}$ and $t_3 = 3$.
$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$. $\mathcal{T}_{13} = \{E\}$ and $t_{13} = 1$.
$\mathcal{T}_{23} = \{F,G,H,I\}$ and $t_{23} = 4$.

Solution (c) for Condition 2 when $m = 5$



Treatment A occurs on all pairs involving individual 1.
Each other treatment is involved with each other individual exactly once.

Equitable partitions

Now we turn to the combinatorics.

Equitable partitions

Now we turn to the combinatorics.
Let Γ be the graph, and Π a partition of its vertex set into t parts $P_1, \ldots, P_t$.

Equitable partitions

Now we turn to the combinatorics.
Let Γ be the graph, and Π a partition of its vertex set into t parts $P_1, \ldots, P_t$.
We say that Π is *equitable* if there is a $t \times t$ matrix $M = (m_{ij})$ of non-negative integers such that each vertex of P_i has exactly m_{ij} neighbours in P_j.

Equitable partitions

Now we turn to the combinatorics.
Let Γ be the graph, and Π a partition of its vertex set into t parts $P_1, \ldots, P_t$.
We say that Π is *equitable* if there is a $t \times t$ matrix $M = (m_{ij})$ of non-negative integers such that each vertex of P_i has exactly m_{ij} neighbours in P_j.
In fact we have been discussing equitable partitions of strongly regular graphs under another name!

Theorem

The treatment allocation to the vertices of a strongly regular graph Γ *given by the partition* Π *has the COBS property (our Condition B) if and only if* Π *is equitable.*

Proof of the theorem

Let Ω be the vertex set of the graph. Let W_0, W_1, W_2 be the common eigenspaces of $A(\Gamma)$, I and J.

Proof of the theorem

Let Ω be the vertex set of the graph. Let W_0, W_1, W_2 be the common eigenspaces of $A(\Gamma)$, I and J.
For $i = 1, \ldots, t$, let V_i be the subspace spanned by the vector which is one on the vertices in part P_i and zero elsewhere, and let V be the space spanned by $V_1, \ldots, V_t$.

Proof of the theorem

Let Ω be the vertex set of the graph. Let W_0, W_1, W_2 be the common eigenspaces of $A(\Gamma)$, I and J.

For $i = 1, \ldots, t$, let V_i be the subspace spanned by the vector which is one on the vertices in part P_i and zero elsewhere, and let V be the space spanned by $V_1, \ldots, V_t$.

The displayed equation near the top of p.144 in the paper by us with Sasha Gavrilyuk and Sergey Goryainov (reference below) shows that, if the partition is equitable, then

$$V = (V \cap W_0) \oplus (V \cap W_1) \oplus (V \cap W_2).$$

But this is equivalent to our Condition B, the definition of a COBS.

Proof of the theorem

Let Ω be the vertex set of the graph. Let W_0, W_1, W_2 be the common eigenspaces of $A(\Gamma)$, I and J.
For $i = 1, \ldots, t$, let V_i be the subspace spanned by the vector which is one on the vertices in part P_i and zero elsewhere, and let V be the space spanned by $V_1, \ldots, V_t$.
The displayed equation near the top of p.144 in the paper by us with Sasha Gavrilyuk and Sergey Goryainov (reference below) shows that, if the partition is equitable, then

$$V = (V \cap W_0) \oplus (V \cap W_1) \oplus (V \cap W_2).$$

But this is equivalent to our Condition B, the definition of a COBS.
The converse works in the same way.

Proof of the theorem

Let Ω be the vertex set of the graph. Let W_0, W_1, W_2 be the common eigenspaces of $A(\Gamma)$, I and J.
For $i = 1, \ldots, t$, let V_i be the subspace spanned by the vector which is one on the vertices in part P_i and zero elsewhere, and let V be the space spanned by $V_1, \ldots, V_t$.
The displayed equation near the top of p.144 in the paper by us with Sasha Gavrilyuk and Sergey Goryainov (reference below) shows that, if the partition is equitable, then

$$V = (V \cap W_0) \oplus (V \cap W_1) \oplus (V \cap W_2).$$

But this is equivalent to our Condition B, the definition of a COBS.
The converse works in the same way.

- R. A. Bailey, Peter J. Cameron, Alexander L. Gavrilyuk and Sergey V. Goryainov, Equitable partitions of Latin-square graphs, *J. Combinatorial Designs* **27** (2019), 142-160; doi: `10.1002/jcd.21634`