Sudoku and mathematics

Peter J. Cameron

Richard and Louise Guy Lecture, Calgary
26 September 2024

What is Sudoku?

8	6		5	7		3		
9		7			6		1	
		5	9	1	8			4
7							8	
1				9				6
	5							7
2			6	8	9	5		
	9		3			6		1
		3		4	5		9	8

Daily Sudoku: Fri 27-Nov-2009

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8	6		5	7		3		
9		7			6		1	
		5	9	1	8			4
7							8	
1				9				6
	5							7
2			6	8	9	5		
	9		3			6		1
		3		4	5		9	8

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Put numbers $1 \dots 9$ in the empty cells so that each number occurs once in each row, once in each column, and once in each 3×3 subsquare.

Sudoku

There's no mathematics involved. Use logic and reasoning to solve the puzzle.

Instructions in *The Independent*

Sudoku

> *There's no mathematics involved. Use logic and reasoning to solve the puzzle.*

Instructions in *The Independent*

But isn't that as good a definition of mathematics as you could get?

Sudoku

> *There's no mathematics involved. Use logic and reasoning to solve the puzzle.*
>
> Instructions in *The Independent*

But isn't that as good a definition of mathematics as you could get?
I suspect that they just said "no mathematics" because you don't have to do arithmetic operations on the numbers in the square (unlike variants such as Kakuro or Killer Sudoku).

Who invented Sudoku?

Was it

- Choi Seok-jeong
- Jacques Ozanam
- Leonhard Euler
- W. U. Behrens
- John Nelder
- Howard Garns
- Wayne Gould
- Robert Connelly

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All of these made some contribution. I will briefly describe some of the history. All of the first three actually considered a slightly different question.

Euler

Euler posed the following question in 1782.

> *Of 36 officers, one holds each combination of six ranks and six regiments. Can they be arranged in a 6×6 square on a parade ground, so that each rank and each regiment is represented once in each row and once in each column?*

He was not the first to consider a similar problem. For example, Ozanam asked whether the sixteen court cards in a pack of cards (jack, queen king and ace) can be arranged in a 4×4 square such that each suit and each value occurs once in each row and column.

Ozanam: Yes

Ozanam gave a solution to his question:

Euler: NO!!

Why was Euler interested?

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Magic squares have interested mathematicians for millennia, going back to the ancient Chinese, who discovered the 3×3 *Luo-Shu* square around the beginning of the current era. They were an active research area for mathematicians of many cultures, including Indian, Greek, Babylonian, Egyptian, and Pre-Columbian American.

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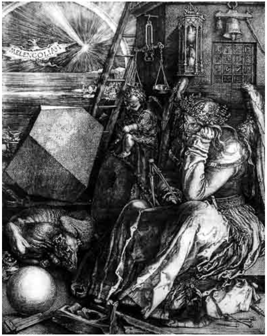
Many people regarded a magic square as a talisman which would keep them safe in battle if they wore it.

Melancholia

Here is Dürer's famous picture *Melancholia* from 1514. Note the mathematical and astronomical items featured.



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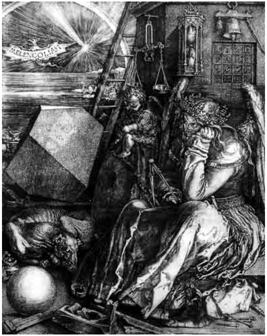

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

I have enlarged the magic square on the wall. All rows, columns and diagonals sum to 34.

Euler's construction

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Suppose we have a solution to Euler's problem with n^2 officers in an $n \times n$ square. Number the regiments and the ranks from 0 to $n-1$; then each officer is represented by a 2-digit number in base n, in the range $0 \ldots n^2 - 1$. Add one to get the range $1 \ldots n^2$. It is easy to see that the row and column sums are constant. A bit of rearrangement usually makes the diagonal sums constant as well.

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$C\beta$	$A\gamma$	$B\alpha$
$A\alpha$	$B\beta$	$C\gamma$
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$C\beta$	$A\gamma$	$B\alpha$
$A\alpha$	$B\beta$	$C\gamma$
$B\gamma$	$C\alpha$	$A\beta$

21	02	10
00	11	22
12	20	01

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21	02	10
00	11	22
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7	2	3
0	4	8
5	6	1

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21	02	10
00	11	22
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7	2	3
0	4	8
5	6	1

8	3	4
1	5	9
6	7	2

Latin squares

A Latin square of order n is an $n \times n$ array containing the symbols $1, \ldots, n$ such that each symbol occurs once in each row and once in each column. If a Graeco-Latin square has two sets of symbols, then a square containing only Latin letters should be a "Latin square"!

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Latin squares have many important uses in mathematics and its applications. Among these are cryptography (one-time pads) and statistics (experimental design).

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Plaintext: $ABCABCABC$

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Substitution table:

	A	B	C
A	B	C	A
B	C	A	B
C	A	B	C

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	A	B	C
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Ciphertext: *ACABACCBBB*

Cryptography, continued

If the encryption table is not a Latin square, then either some entry is repeated in a column (in which case the message fails to be uniquely decipherable), or some entry is repeated in a row (in which case some information is leaked to the interceptor).

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In the Second World War, the Japanese military ciphers often used the digits $0 \cdots 9$ as symbols. The ciphers would also often use a codebook where various commonly used terms were encoded as groups of four digits. Thus, for example, `0700` could refer to the *kōkū tokushi musentai* (Air Special Radio Unit), and `4698` to the *kōkū tokushu jōhōtai* (Air Special Intelligence Unit). The key was a string of pseudo-random digits.

Substitution square for Japanese 6633 cipher

	0	1	2	3	4	5	6	7	8	9
0	4	9	5	3	2	7	0	1	6	8
1	7	5	0	9	3	2	1	8	1	4
2	3	1	7	2	8	0	9	6	9	7
3	0	8	4	7	0	1	3	4	5	2
4	5	3	2	4	9	3	8	2	7	6
5	9	0	1	6	7	5	4	7	2	3
6	2	6	8	0	0	9	7	5	3	1
7	6	2	6	1	4	8	6	0	8	5
8	1	7	9	7	1	4	5	9	0	7
9	8	4	3	5	5	6	2	3	4	0

Spot the flaw!

Orthogonal Latin squares

Two Latin squares A and B are orthogonal if, given any k, l, there are unique i, j such that $A_{ij} = k$ and $B_{ij} = l$. Thus, a Graeco-Latin square is a pair of orthogonal Latin squares.

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But apart from that he was completely wrong. In 1960, Bose, Shrikhande and Parker (the "Euler spoilers") showed that there is a pair of orthogonal Latin squares for every order but 2 and 6.

Latin squares in statistics

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A Latin square at Rothamsted Experimental Station, designed by R. A. Bailey; thanks to Sue Welham.

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Next year, we want to run another experiment on the same trees. But the trees will still carry the effects of the previous treatments. So we should use a Latin square orthogonal to the previous one.
Moral: Use a Latin square which has an "orthogonal mate". (Not all of them do!)

Behrens

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Take an $n \times n$ grid divided into n regions, with n cells in each. A gerechte design for this partition involves filling the cells with the numbers $1, \ldots, n$ in such a way that each row, column, or region contains each of the numbers just once. So it is a special kind of Latin square.

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Example

Suppose that there is a boggy patch in the middle of the field.

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

Orthogonal mates are gerechte designs

Given a Latin square, the positions of the n symbols give rise to n "regions" of the grid.

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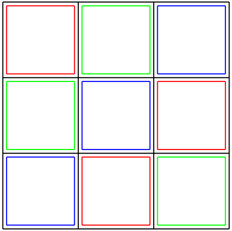

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3	1	2

1	2	3
3	1	2
2	3	1

Nelder

The statistician John Nelder defined a critical set in a Latin square in 1977. This is a partial Latin square which can be completed in only one way.

Example

1		
	2	

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Example

1		
	2	

1	3	
3	2	

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Example

1		
	2	

1	3	
3	2	

1	3	2
3	2	1
2	1	

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Example

1		
	2	

1	3	
3	2	

1	3	2
3	2	1
2	1	

1	3	2
3	2	2
2	1	3

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Example

1		
	2	

1	3	
3	2	

1	3	2
3	2	1
2	1	

1	3	2
3	2	2
2	1	3

A trade in a Latin square is a collection of entries which can be "traded" for different entries so that another Latin square is formed. For example, two rows of a Latin square form a trade, since we can simply swap them to get a different Latin square.

Trades and critical sets

A subset of the entries of a Latin square is a critical set if and only if it meets every trade.

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For example, a critical set can miss at most one row, at most one column, and at most one symbol (as Sudoku solvers know!).
What is the size of the smallest critical set in an $n \times n$ Latin square? Nelder conjectured that the answer is $\lfloor n^2/4 \rfloor$. This is true for $n \leq 8$, but Keith Hermiston found an improved lower bound for $n \geq 9$.

Garns

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Garns called his puzzle "number place". It became popular in Japan under the name "Sudoku" in 1986 and returned to the West some years later, popularised at first by the New Zealander Wayne Gould.

Connelly

Robert Connelly proposed a variant which he called symmetric Sudoku. The solution must be a gerechte design for all these regions:

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

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2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Rows

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3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Columns

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4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Subsquares

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3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Broken rows

Connelly

Robert Connelly proposed a variant which he called symmetric Sudoku. The solution must be a gerechte design for all these regions:

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Broken columns

Connelly

Robert Connelly proposed a variant which he called symmetric Sudoku. The solution must be a gerechte design for all these regions:

3	5	9	2	4	8	1	6	7
4	8	1	6	7	3	5	9	2
7	2	6	9	1	5	8	3	4
8	1	4	7	3	6	9	2	5
2	6	7	1	5	9	3	4	8
5	9	3	4	8	2	6	7	1
6	7	2	5	9	1	4	8	3
9	3	5	8	2	4	7	1	6
1	4	8	3	6	7	2	5	9

Locations

In stained glass

Here is the same square done in stained glass (with colours for numbers) by David Spiegelhalter, the Professor of Public Understanding of Risk at the University of Cambridge.

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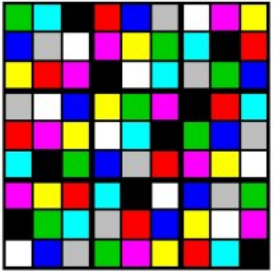

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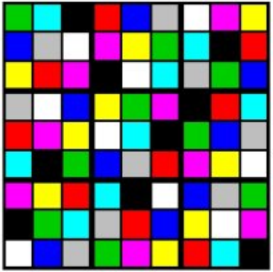

A question for you: Is it art?

A symmetric Sudoku puzzle

Here for your entertainment is a puzzle which is to be completed using the rules just described for symmetric Sudoku.

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Here for your entertainment is a puzzle which is to be completed using the rules just described for symmetric Sudoku.

								7
				7				
		6						
		4		3				
			1	5				8
					2		7	
					1	4		
					4			
1								

All symmetric Sudoku solutions

Symmetric Sudoku has a beautiful mathematical structure.

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Symmetric Sudoku has a beautiful mathematical structure.
Rosemary Bailey, Bob Connelly and I had a paper in the *American Mathematical Monthly* in 2008 showing that there are just two essentially different solutions.
The proof involved various topics from mathematics and beyond, such as affine geometry, perfect error-correcting codes, and the card game SET®.
I will say a brief word about how SET enters the discussion.

Sudoku and SET

The card game SET has 81 cards, each of which has four attributes taking three possible values (number of symbols, shape, colour, and shading). A winning combination is a set of three cards on which either the attributes are all the same, or they are all different.

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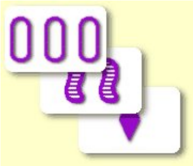 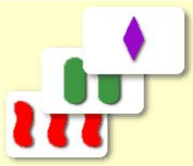

Sudoku, SET, and geometry

Each cell of the Sudoku puzzle corresponds to one SET card, since it can be described by four attributes each with three values (the "large row" containing it, the row within the "large row", the "large column", and the column within the "large column". Connelly's six types of region have simple descriptions in this representation.

Sudoku, SET, and geometry

Each cell of the Sudoku puzzle corresponds to one SET card, since it can be described by four attributes each with three values (the "large row" containing it, the row within the "large row", the "large column", and the column within the "large column". Connelly's six types of region have simple descriptions in this representation.
Also, if you know some geometry, we can number the three attributes by the set $\mathbb{F}_3$ of integers mod 3. So, for example, the point $(1, 2, 0, 1)$ labels the cell in row 6 and column 2.
The lines in this geometry are the sets of points of the form $a + bt$, where a and b are fixed and t runs through $\mathbb{F}_3$. They are precisely the winning combinations in SET!

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> *the number of orbits of a group on a finite set is equal to the average number of fixed points of the group elements.*

An earlier computation by Felgenhauer and Jarvis gives the total number of solutions to be 6 670 903 752 021 072 936 960. Now for each conjugacy class of non-trivial symmetries of the grid, it is somewhat easier to calculate the number of fixed solutions.

Some things I haven't told you

- How many Sudoku puzzles are there? How many are critical sets, in the sense that if we leave out any entry then the solution is no longer unique?

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- What is the smallest number of entries in a Sudoku critical set?

The first is not known. But the second is . . .

Smallest number of entries in a Sudoku puzzle

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Here is a 17-entry puzzle constructed by Gordon Royle.

							1	
4								
	2							
				5		4		7
		8				3		
		1		9				
3			4			2		
	5		1					
			8		6			