Inverse group theory

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(with J. Araũjo, R. A. Bailey, C. Casolo, D. Craven, H. R. Dorbidi, M. Giudici, S. Harper, F. Matucci, C. Quadrelli, G. Royle, B. Sambale *et al.*)

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Why so many coauthors?

The Argentine author Jorge Luis Borges wrote an essay entitled "Kafka and his precursors". He pointed to many works of literature, from cultures all over the world, which we would not have regarded as part of a single genre had Franz Kafka not existed. As it is, we would call all these works "Kafkaesque".

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Which groups are integrable?

Embarrassingly, we don't know; we don't even know if it is decidable for finite groups…

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This is a completely satisfactory solution to the question, but it leaves open various other questions, such as: how many groups of order n are Frattini subgroups? For which n is every group of order n a Frattini subgroup? And what happens for infinite groups?

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- An abelian group of order n has an integral of order at most $n^{1+o(1)}$, but may fail to have an integral of order bounded by cn for constant c.
- A finite group can be integrated n times (in the class of finite groups) for every n if and only if it is a central product of an abelian group and a perfect group.

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Now we feel it is time to turn to the more general problem of inverse group theory.

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- What can be said about all such H, for some fixed G?

As we have seen, the problem is completely solved for the Frattini subgroup of a finite group. But for the derived group, we do not even know whether it is decidable!

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- The derived quotient of any group is abelian. But every abelian group is the derived quotient of some group (namely, itself).

These examples require no further comment. In the rest of the talk I describe a small selection of more interesting cases.

Schur multiplier

Recall that the Schur multiplier $M(G)$ of the finite group G is the (unique) largest abelian group Z for which there exists a group H with $Z \leq Z(H) \cap H'$ and $H/Z \cong G$. There are of course many other definitions.

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Proof.

A theorem of Schur says that

$$M(G \times H) = M(G) \times M(H) \times (G \otimes H).$$

Now $G \otimes H$ vanishes if G and H are perfect, so it is enough to realise arbitrary cyclic groups as Schur multipliers of perfect groups. Now C_n is the Schur multiplier of $\mathrm{PSL}(n, p)$ if $p \equiv 1 \pmod{n}$ with a few small exceptions; and Dirichlet's theorem guarantees infinitely many such primes p. □

Derangements

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Frobenius complements have a very restricted structure, as shown by Zassenhaus in the 1930s. We did find others: the Klein group V_4 and the alternating groups A_4 and A_5; these are derangement quotients of groups of degrees 5^4, 23^4 and 59^4 respectively.

Varieties

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Also, if G has a verbal inverse, then it has a finitely generated verbal inverse.

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This fails for the variety of nilpotent groups of class 2. Perhaps it is true only for abelian varieties.

An inverse inverse problem

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Note that this proposition does *not* cover the Frattini subgroup.

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Our bounds, roughly p^{cm^2} and p^{p^m}, are a long way apart!

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For example, a group G has order divisible by 6 if and only if it embeds one of the groups C_6, S_3, and A_4.

But that is the end of my story.

… for your attention.

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