Graphs associated with groups

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- the automorphism group, endomorphism monoid, or graphs built from groups such as Cayley graphs.

I will talk mostly about the second of these topics.

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We would expect to find that graphs associated with algebraic structures are less scruffy than general graphs. Later I will show you some examples of beautiful graphs from groups.

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- Graphs built on groups reflecting the group structure.

I will spend most time on the third topic, where my current interest lies. But you should be aware that Cayley graphs represent the largest and most significant of the three topics.

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When the axioms for a group were written down by Dyck in the later part of the nineteenth century, group theory was a hundred years old. So that the researches of Galois, Cauchy, Jordan and others should not be lost, it was necessary to show that the two concepts agree. Any transformation group satisfies the group axioms (composition of mappings is always associative, and the other three axioms are required by the definition). So it remained to show that every "abstract" group can be represented as a permutation group, and this is what Cayley did.

Cayley's theorem

Given an abstract group G (a structure satisfying the group axioms), Cayley builds a permutation group as follows. The domain V is just the set G. For each $g \in G$, define a map $\pi_g : V \to V$ by the rule

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So any Cayley digraph admits the regular representation of G as a group of automorphisms.

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- $\mathrm{Cay}(G, S)$ is an undirected graph if $s \in S \Rightarrow s^{-1} \in S$.
- $\mathrm{Cay}(G, S)$ is an oriented graph if $s \in S \Rightarrow s^{-1} \notin S$.

Graphical regular representations

Much of the interest in finite Cayley graphs has been in the question:

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This is about restricting the symmetry of a graph. But I like symmetry …

Something different

Let $G = (C_2)^4$, where the four factors are generated by e_1, e_2, e_3, e_4. Let $\Gamma = \text{Cay}(G, S)$, where $S = \{e_1, e_2, e_3, e_4, e_1e_2e_3e_4\}$.

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It has a high degree of symmetry. The stabiliser of a vertex acts as the symmetric group S_5 on its neighbours; the full group is $(C_2^4) \rtimes S_5$, of order 1920.

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It is not too hard to show that, up to isomorphism, it is the unique strongly regular graph with these parameters.

Algebraic theory

It is not hard to see that the adjacency matrix A of a strongly regular graph Γ (the matrix with rows and columns indexed by vertices, with (v, w) entry 1 if v is joined to w and 0 otherwise) satisfies

$$A^2 = kI + \lambda A + \mu(J - I - A),$$

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Solving this quadratic and computing the multiplicities of the eigenvalues gives strong necessary conditions on (n, k, λ, μ).

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At the conference dinner that night, Donald Higman and Charles Sims speculated on whether there might be another simple group acting on a 100-vertex graph. After dinner, they went off to see what they could come up with. Higman was familiar with the theory of strongly regular graphs, and knew where to look. By the end of the evening, they had constructed their group, using familiar properties of the Witt design, the Steiner system on 22 points with automorphism group $M_{22} \rtimes C_2$.

The Higman–Sims graph

The construction is simple. The vertices are the 22 points and 77 blocks of the Witt design, and one further vertex $*$. Now $*$ is joined to all the points; a point and block are joined if they are incident; and two blocks are joined if they are dsjoint.

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The graph is strongly regular, with parameters $(100, 22, 0, 6)$.

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One of these was Dale Mesner. We later learned that he had constructed the Higman–Sims graph twelve years earlier. But he had a bigger struggle, since he didn't know about the Witt design; and being a statistician, not a group theorist, he never thought to investigate the autmorphism group of the graph he had found.

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I suggest that we refer to the graph as the Mesner graph, but continue to call the group the Higman–Sims group.

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There is no obvious reason why there can't be any more, but all attempts to construct one have so far failed.

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In the case of finitely generated groups, where we can take the Cayley graphs to have finite valency, we are in the realm of geometric group theory. For many groups, "triangles in the Cayley graph are narrow": given three points a, b, c, geodesics from a to b and from b to c do not stray too far from the geodesic from a to c.

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In the case of finitely generated groups, where we can take the Cayley graphs to have finite valency, we are in the realm of geometric group theory. For many groups, "triangles in the Cayley graph are narrow": given three points a, b, c, geodesics from a to b and from b to c do not stray too far from the geodesic from a to c.

Such groups are called hyperbolic, and feature in the work of Gromov and many others.

Random Cayley graphs

What if we allow the connection set to be infinite?

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In the 1960s, Erdős and Rényi proved the remarkable result that, if we take a countable set of vertices, and choose edges at random (by tossing a coin, say), then there is one particular graph R which will arise almost surely. This is the Erdős–Rényi random graph, sometimes called the Rado graph (since Rado gave the first explicit construction of it).

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It is known that for a large class of countably infinite groups G (including all abelian groups of infinite exponent), if we choose a random connection set S (by including inverse pairs s, s^{-1} in S at the toss of a coin), the graph $\mathrm{Cay}(G, S)$ is isomorphic to R. So R is a Cayley graph for many different groups!

Some more history

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Much of the work on the Classification involves determining all simple groups with a given involution centraliser; indeed, several sporadic groups were found this way.

Enter the commuting graph

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The commuting graph is just the first of a number of graphs defined on groups, which form the topic I will turn to now. These include the power graph, enhanced power graph, nilpotency graph, solubilty graph, Engel graph, and generating graph.

3-transposition groups

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The classic example is the symmetric group $G = S_n$, where C is the conjugacy class of transpositions. In this case, xy has order 3 if and only if the 2-element supports of x and y have non-trivial intersection. So the graph is the famous strongly regular triangular graph, also known as the line graph of the complete graph K_n.

Classification

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Jonathan Hall weakened Fischer's assumptions and extended the result to the infinite. There are also connections with Moufang loops (in the work of Yuri Manin) and finite geometry.

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I was able to prove that the power graph determines the directed power graph, at least up to isomorphism.

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Over the next couple of decades, I wrote a couple of papers on the power graph, but this was not my main focus. Then someone asked me a question. I do not now remember who it was, or what the question was, but it caught my interest.

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Then two things happened. First, Alireza Abdollahi from Isfahan, the editor-in-chief of the *International Journal of Group Theory*, saw it, and invited me to submit it to his journal. Since it is a free open access journal, which I like, I was happy to agree. (It is now the most highly cited paper in the journal.)

The graphs-on-groups virus hits India

Then Ambat Vijayakumar from Kochi saw it, and invited me to lead an on-line research discussion. This was in 2021, at the height of the Covid pandemic. I do believe this helped to keep me sane at that difficult time. It ran for the whole summer, and many new results were presented and new projects started.

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Many graph theorists, meeting a new graph, enquire into its properties and parameters: Is it Hamiltonian? Regular? What is its clique number, energy, matching number, etc.? To me, this is interesting, but my main concern is the interplay between graphs and groups, so I will start off by outlining my philosophy on this.

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- We may find beautiful and interesting graphs in the process. But we may have to strip away some uninteresting stuff first.

Dominating or isolated vertices

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For example, the set of dominating vertices in the commuting graph of a group is the centre of the group,

$$Z(G) = \{g \in G : (\forall x \in G)(gx = xg)\}.$$

If we don't remove them, then the graph will be connected with diameter at most 2, since we can get from x to y in two steps via any vertex in the centre.

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We will see various examples later.

Meet the cast

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- The complete graph.

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Groups with all elements of prime power order are called *EPPO groups*, and have quite a long history. Higman found the soluble ones in the 1950s, and Suzuki the simple ones in the 1960s. In 1981 Brandl gave the complete classification; but it was published in a rather obscure Italian journal, and so was rediscovered a couple of times.

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Groups containing no subgroup $C_p \times C_p$ have Sylow subgroups which are cyclic or generalized quaternion. Now theorems of Burnside, Glauberman, and Gorenstein and Walter give us a result from which the classification can be read off.

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- The rank graph, where x and y are joined if $\{x, y\}$ is contained in a generating set of minimum size (this minimum size is the rank of G).

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Proof of (b): if x and y are powers of z, and a generating set contains both x and y, we can get a smaller one by deleting x and y and including z.
Proof of (c): clear.

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Equality in the three cases of (c) is realised by minimal non-abelian (resp., minimal non-nilpotent, minimal non-soluble) groups. In the first two cases there are complete classifications known. In the third, a minimal non-soluble group has a minimal simple group as a quotient, and such groups are all known (using Thompson's work on N-groups); this is enough for many purposes.

Graphs and equivalence relations

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Here a Dedekind group is a group in which every subgroup is normal (these are all known).
A 2-Engel group is one satisfying the identity $[[x,y],y] = 1$, where $[x,y]$ is the commutator $x^{-1}y^{-1}xy$. The class of 2-Engel groups is contained in the class of nilpotent groups of class 3, and contains the class of nilpotent groups of class 2.

Compressed graphs

In the situation where we have a graph and an equivalence relation on a group, it is sometimes more convenient to contract each equivalence class to a single vertex. So there is an edge between classes C_1 and C_2 in the compressed graph if and only if there exist $x_1 \in C_1$ and $x_2 \in C_2$ such that there is an edge from x_1 to x_2 in the original graph.

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These were studied independently of the super graphs on the last slide, and typically go by different names. For example, the commuting conjugacy class graph, or CCC-graph of a group G has vertices the conjugacy classes in G, with C_1 and C_2 joined if there exist $x_1 \in C_1$ and $x_2 \in C_2$ such that $x_1 x_2 = x_2 x_1$.

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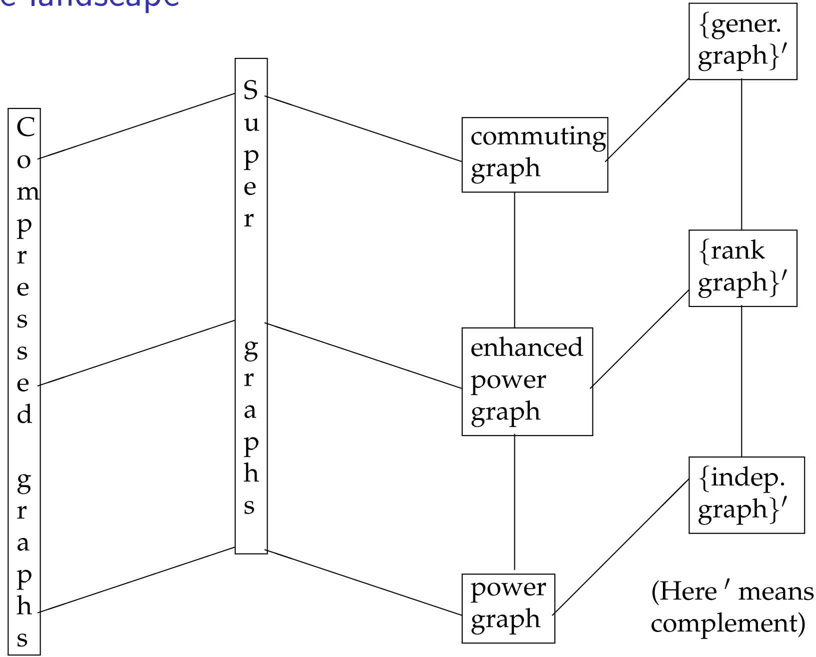
Similarly, we have the nilpotent and soluble conjugacy class graphs, where we join C_1 to C_2 if there exist $x_1 \in C_1$ and $x_2 \in C_2$ such that $\langle x_1, x_2 \rangle$ is nilpotent (resp. soluble). These are called the NCC-graph and SCC-graph, for short.

The landscape



(Here ′ means complement)

On a theorem of Landau

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Unlike Landau, we use the Classification of Finite Simple Groups, but only in a rather low-key way, and we suspect that this can be avoided.

Cliques in the power graph and enhanced power graph

If some elements of a group are such that any pair generate a cyclic group, then all of them lie in a cyclic group. So a maximal clique in the enhanced power graph is a maximal cyclic subgroup, and the clique number is the largest order of an element of the group.

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A clique in the power graph is contained in a clique in the enhanced power graph, so a maximal clique is contained in a cyclic subgroup, but not necessarily one of maximum order.

For the power graph of a cyclic group of order n has order $F(n)$, where F is the arithmetic function defined by

$$F(1) = 1, F(n) = F(n/p) + \phi(n) \text{ for } n > 1,$$

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where ϕ is Euler's function and p is the smallest divisor of n. For example, in the group $PGL(2, 11)$, the maximum element orders are 10, 11 and 12, so the enhanced power graph has clique number 12. But $F(10) = 9$, $F(11) = 11$, and $F(12) = 9$, so the power graph has clique number 11.

Cliques in the power graph

The recurrence for F can be used to show that $\phi(n) \leq F(n) \leq 3\phi(n)$. In fact,

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Theorem

The enhanced power graph of a finite group is weakly perfect (that is, has clique number equal to chromatic number).

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We saw earlier that we can get interesting graphs by restricting to a conjugacy class. (So the induced subgraph of the commuting graph of S_n on the class of transpositions is the complement of the line graph of K_n.)
But can we get the graph itself to tell us where the jewel lies?

A pointer

Ten years ago, Colva Roney-Dougal and I looked at the automorphism group of the generating graph of the alternating group A_5 (a group of order 60). We expected that the result would be the automorphism group of A_5 (which is S_5), or something near that.

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Where does the rubbish come from, and how do we get rid of it?

Twins

Two vertices v, w of a graph are twins if they have the same neighbours apart possbily from one another; that is, they are not joined and have the same open neighbourhoods, or they are joined and have the same closed neighbourhoods. So there are two kinds of twins, open and closed; but the distinction won't concern us.

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These local automorphisms generate a subgroup which is a direct product of symmetric groups, but tells nothing about the graph structure except which pairs of vertices are twins.

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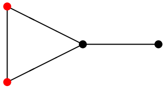

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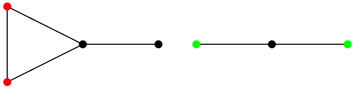

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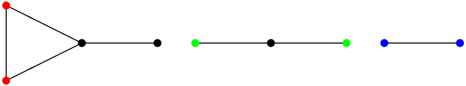

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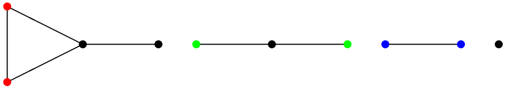

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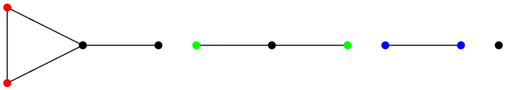

Theorem
Two graphs obtained by twin reduction of the same graph are isomorphic.

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The concept arose many times and was given different names by different authors, including "complement-reducible graph" and "hereditary Dacey graph".

Theorem

The cokernel of a graph Γ *is a single vertex if and only if* Γ *is a cograph.*

When is a graph on a group a cograph?

This gives point to the question:

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We found four types of behaviour (more are possible).

Cokernels of the difference graph on simple groups

- The simplest is the class of EPPO groups where, as we saw, the power graph and enhanced power graph are equal, so that $D(G)$ has no edges. As we saw, these groups were determined by Brandl. Among simple groups we have a small finite number of groups $\mathrm{PSL}(2,q)$ and $\mathrm{Sz}(q)$ and the group $\mathrm{PSL}(3,4)$.

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- Next come the groups whose difference graph is a cograph, so that the cokernel has just a single vertex. We determined the simple groups for which this condition holds. We get a few more groups $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$.

- Next come groups where the cokernel of the difference graph consists of a number of small components, pairwise isomorphic. For the groups $\mathrm{PSL}(2,23)$ and $\mathrm{PSL}(2,25)$, we obtain respectively 253 or 325 copies of the graph $K_5 - P_4$. We do not know why the components are the same in the two cases.

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I will tell you about a couple of these.

Some beautiful graphs

- $G = M_{11}$, the smallest Mathieu group. In this case, we obtain a bipartite graph on $165 + 220$ vertices; it is semiregular, with valencies 4 and 3 in the two bipartite blocks; it is connected; and it has diameter 10 and girth 10. Its automorphism group is M_{11}.

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- $G = \mathrm{PSL}(3, 3)$. This acts on the projective plane over the field with 3 elements. The cokernel of the difference graph has 169 vertices, which can be identified with the point-line pairs in the plane; these fall into two types, flags and antiflags of the graph, which are bipartite blocks, with valencies 9 and 4. The graph is connected with diameter 5 and girth 6. Its automorphism group is $\mathrm{PGL}(3, 3)$.

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The first is clearly "sporadic", but the second can be defined over any finite field, and might be an interesting topic for finite geometers to study.

Simplicial complexes

A simplicial complex Δ is a downward-closed collection of finite subsets (called simplices or simplexes) of a set X. We assume that every singleton of X belongs to Δ. For geometric reasons, a simplex of cardinality k has dimension $k-1$. Thus a point or vertex of X has dimension 0, while an edge $\{x, y\}$ has dimension 1, and a triangle $\{x, y, z\}$ has dimension 2.

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It seems that there are many ideas and problems about graphs on groups which have analogues for simplicial complexes. I will describe just two simplicial complexes on a group, and pose a problem connecting them.

Independence and strong independence

A subset A of a group G is independent if none of its elements can be expressed as a word in the other elements and their inverses; equivalently, if $a \notin \langle A \setminus \{a\} \rangle$ for all $a \in A$. The independence complex consists of all the independent subsets of G. It is a simplicial complex.

A subset A of a group G is called strongly independent if no subgroup of G containing A has fewer than $|A|$ generators. The strong independence complex of G is the complex whose simplices are the strongly independent sets. It is a ls a simplicial complex.

Two problems

Question

For which groups G do the independence and strong independence complexes of G coincide?

It is known that being an EPPO group (all elements have prime power order) is a necessary condition, while being an abelian p-group, for prime p, is sufficient.

Question

For which groups G do the simplexes of maximal cardinality in the independence complex generate G?

This is the case for the symmetric group S_n, by a theorem of Julius Whiston, who also showed that the maximum cardinality is $n-1$.