The Road Closure Problem

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Urals Seminar on Group Theory and Combinatorics
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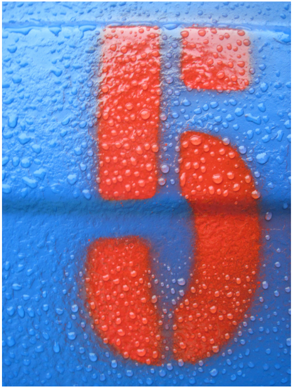

years of UWGTC

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Everything in this talk will be finite.

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I will discuss briefly the situation for the property of regularity before turning to the property that leads us to Road Closure.

Regularity and the universal transversal property

A semigroup S is regular if every element has a quasi-inverse y satisfying $xyx = x$. Note that we can assume in addition that $yxy = y$. For, if $xyx = x$ and $z = yxy$, then

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Theorem

A permutation group G has the ut property if and only if, for every non-permutation f, the semigroup $\langle G, f\rangle$ is regular.

Groups with the ut property

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In their book on the theory of games, von Neumann and Morgenstern asked for a classification of these groups. In the second edition, they claim that such a classification was given by Chevalley (I have not seen this). Wielandt attributes the result to Beaumont and Peterson. The conclusion is that they are symmetric or alternating groups together with four small exceptions of degrees 5, 6, 9 and 9.

The k-universal transversal property

Let k be a positive integer with $k \leq |\Omega|/2$. A permutation group G on Ω has the k-universal transversal property (for short, the k-ut property) if for every subset A of Ω and partition P of Ω with $|A| = |P| = k$, there is an element of G which maps A to a transversal of P.

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Thus k-ut has something in common with k-set transitivity. We note in particular that a k-set transitive group has k-ut, since given A and P we select a transversal B for P and map A to B.

k-ut in detail

For $2 \leq k \leq n/2$, the k-set transitive groups are completely known. In the early 1970s, Kantor showed that they are k-transitive, with known exceptions; and the classification of k-transitive groups for $k \geq 2$ follows from the Classification of Finite Simple Groups. In particular, a 6-set transitive group of degree at least 12 is symmetric or alternating

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Similarly, we were able to show that, for $6 \leq k \leq n/2$, a group with k-ut is symmetric or alternating; for $k = 5$, such a group is either 5-transitive or $\mathrm{P\Gamma L}(2, 32)$. For $k = 3$ and $k = 4$, there are some families of groups for which the status of k-ut is not known. One of these families consists of the Suzuki groups $\mathrm{Sz}(q)$, for which we don't know if they have 3-ut. (Pablo Spiga recently found a computer-free proof that $\mathrm{Sz}(8)$ has 3-ut; Leonard Soicher has checked this with GAP and shown the same for $\mathrm{Sz}(32)$.)

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But for $k = 2$, things are very different …

2-ut and primitivity

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Conversely, if G has an invariant partition P, and x, y lie in the same part, then no translate of $\{x, y\}$ is a transversal to the partition one of whose parts is a part of P. □

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Since G has k-ut, there is an element $g \in G$ such that Ag is a transversal to P. Then gf induces a permutation on A, so there exists a number m such that $(gf)^m$ is an idempotent. □

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If you know what an independence algebra is, you will probably guess that the result extends to these.

Idempotent generation in $\langle G, f \rangle$

Now semigroups of the form $S = \langle G, f \rangle$ are never going to be idempotent-generated. For a generating set for S must contain a generating set for G (since a permutation cannot be a product including non-permutations), but the only idempotent in G is the identity.

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Groups with the id property were determined by Araújo, Mitchell and Schneider in the same paper I mentioned earlier. Apart from the symmetric and alternating groups, there are only three exceptional groups, with degrees 5 and 6.

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There is also a strengthening of k-ut known as strong k-ut, which implies k-id. So the groups we are after are sandwiched between k-ut and strong k-ut.

Some results

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- $\mathrm{PGL}(2, 2^{23})$ fails 4-ut and 4-id.

The Suzuki groups remain as a stumbling block for both 3-ut and 3-id, though there is recent computational progress by Leonard Soicher.

The Road Closure Property

Let G be a transitive permutation group on Ω. We say that G has the Road Closure Property (RCP) if, for any orbital graph $(\Omega, \mathcal{O})$ and any block of imprimitivity B for the action of G on $\mathcal{O}$, the graph $(\Omega, \mathcal{O} \setminus B)$ is connected.

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In other words, if the orbital graph is thought of as a road network, and workmen dig up some of the roads forming a block of imprimitivity, it is still possible to get between any two vertices in the network by road.

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The name was given when many of the roads in the neighbourhood of St Andrews were closed for extended periods while the University installed a pipe to bring hot water from a boiler in Guardbridge to heat the University buildings.

Connection with idempotent generation

The RCP is the link between the 2-id property and more familiar permutation group properties. João Araújo and I were able to prove, by quite a long argument:

Theorem

A transitive permutation group G on Ω has the 2-id property if and only if it has the Road Closure Property.

We already saw that the 2-ut property is equivalent to primitivity, which itself is equivalent to the connectedness of every orbital graph. As we also saw, the 2-id property is a strengthening of the 2-ut property, and the RCP is clearly a strengthening of the connectedness of all the orbital graphs.

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From 2-id to the RCP

Let G be a permutation group on Ω; let A be a k-subset and P a k-partition of Ω. The Houghton graph associated with the data (G, k, A, P) is the bipartite graph whose vertex set is $AG \cup PG$, with an edge from $A' \in AG$ to $P' \in PG$ if A' is a transversal to P'.

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Now we can show that connectedness of all k-Houghton graphs is a necessary condition for the k-id property; moreover, for $k = 2$, these are equivalent.

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The 2-Houghton graphs potentially have exponentially many vertices. By contrast, the orbital graph (Ω, AG) for a 2-set A has only $n = |\Omega|$ vertices and $O(n^2)$ edges, so is more manageable.

From 2-id to the RCP

Let G be a permutation group on Ω; let A be a k-subset and P a k-partition of Ω. The Houghton graph associated with the data (G, k, A, P) is the bipartite graph whose vertex set is $AG \cup PG$, with an edge from $A' \in AG$ to $P' \in PG$ if A' is a transversal to P'.

Now we can show that connectedness of all k-Houghton graphs is a necessary condition for the k-id property; moreover, for $k = 2$, these are equivalent.

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So the last step is to show that, if a 2-Houghton graph is disconnected, then the corresponding orbital graph gives a failure of the RCP, and conversely. This completes the proof.

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The problem is not completely solved, but we have at least a satisfactory reduction to the almost simple case, together with a number of examples.

The O'Nan–Scott Theorem

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A Cartesian structure on Ω is an identification of Ω with the set of words of length m over an alphabet of size s, for some $m, s > 1$. Its automorphism group is the wreath product $S_s \wr S_m$, in its product action.

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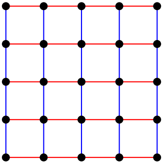
Theorem (O'Nan–Scott)

A primitive basic permutation group is affine, diagonal or almost simple.

Non-basic groups fail the RCP



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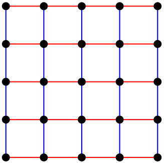

The RCP is closed under taking supergroups, so it suffices to show that the wreath product $S_s \wr S_m$ fails the RCP. For this group, the Hamming graph (in which two m-tuples over an s-letter alphabet are joined if they differ in just one coordinate) is an orbital graph. The m coordinates give blocks of imprimitivity for the group action on edges; the ith block consists of tuples which differ only in the ith coordinate. If we delete this block, we cannot move from a tuple to another with a different value in the ith coordinate.

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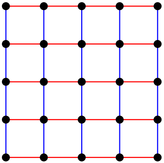

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In the figure, deleting the blue edges leaves a graph whose connected components are the rows.

Affine groups

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A permutation group G on Ω is affine if it has a regular normal subgroup N which is elementary abelian; so we can identify Ω and N with a finite vector space $V = V(d, p)$ and the stabiliser of 0 with a linear group H on V (a subgroup of $\mathrm{GL}(d, p)$). So G is the semidirect product of V by H. Note that, by definition, affine groups are transitive.

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- *If H is an imprimitive linear group, then G is non-basic.*
- *If H is a primitive linear group, then G satisfies RCP.*

This completely settles the question for affine groups.

Diagonal groups

A diagonal primitive group G has a normal subgroup isomorphic to T^{d+1}, acting on the cosets of its diagonal subgroup, where T is a non-abelian finite simple group. Thus the maximal diagonal groups have the form $G \leq T^{d+1}.(\mathrm{Out}(T) \times S_{d+1})$. Here $\mathrm{Aut}(T)$ acts on T^{d+1} by acting in the same way on each coordinate, but the inner automorphisms are induced by the diagonal subgroup so are in T^{d+1}; and the symmetric group S_{d+1} permutes the coordinates. The point stabiliser is $\mathrm{Aut}T \times S_{d+1}$.

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Now let H be a subgroup of S_{d+1}. It is known that the diagonal group $T^{d+1}.(\mathrm{Out}(T) \times H)$ is primitive if and only if H is primitive (where, confusingly, the second occurrence of the word "primitive" here includes the trivial group on two points, since it fixes no non-trivial partition.) To avoid this complication we assume that $d > 1$.

Diagonal groups and RCP

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So we can construct infinitely many groups failing the RCP by starting with a single one and plugging it into this construction, and then iterating. (The degrees of groups in such a sequence grow exponentially!) The starting group could be non-basic, or one of the almost simple examples we will meet next. So although there are infinitely many examples, the situation is well under control for diagonal groups.

Almost simple groups: examples

Theorem

Let G be a primitive group having an imprimitive subgroup N of index 2. Then G fails the RCP.

Proof.

Let P be an N-invariant partition. It is not G-invariant, so has an image Q under $G \setminus N$. Now the meet of P and Q is G-invariant, and so trivial. Call the elements of P "points" and those of Q "blocks", a point and block being "incident" if they intersect in an element of Ω. We obtain an incidence structure on which G acts flag-transitively, and the original domain Ω is identified with the set of flags.

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Now there is an orbital graph for G whose edges are pairs of flags sharing a point or a block. The two types are blocks of imprimitivity, and removing one type disconnects the graph. (If we remove the edges corresponding to flags sharing a block, then we cannot move from a flag to one with a different point. □

Duality and triality

This gives us large numbers of examples failing RCP. But there are more. The incidence structure may be a nice self-dual geometric object such as point-hyperplane pairs in projective space, or not. A family of examples is given by the groups $\mathrm{PGL}(2, p)$, where p is a prime congruent to ± 11 or $\pm 19 \pmod{40}$, acting on the cosets of a subgroup isomorphic to A_5.

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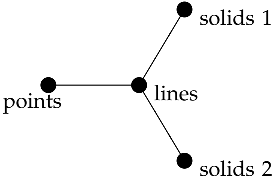

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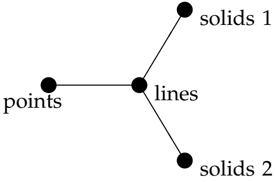

There are just two more known examples where the group has three blocks of imprimitivity on edges of an orbital graph.

Almost simple groups: a conjecture

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The story continues …