Inverse group theory

Peter J. Cameron
University of St Andrews

Péter Pál Pálfy birthday colloquium
Budapest, 29 September 2025

Happy birthday, P^3!

More than half our lives ago, we were co-authors of a paper which turned out to be quite influential, not only for finite groups but for profinite groups too. This was even before we first met (if my memory is correct).

- L. Babai, P. J. Cameron and P. P. Pálfy, On the orders of primitive groups with bounded non-abelian composition factors, *J. Algebra* **79** (1982), 161–168; doi: `10.1016/0021-8693(82)90323-4`

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For your birthday I would like to present another area for which both finite and infinite groups are relevant. So here is a guided tour through quite a bit of group theory.

Integrals of groups

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Now I want to ask a simple question, to convince you that this is more serious than it looks.

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For, for $n = 1, 2, \ldots$, we could determine the groups of order $n|G|$ and check whether any of them has derived group isomorphic to G. A positive answer to the second question above would guarantee that the algorithm terminates.

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I should note, though, that the analogous result for the derived group is false.

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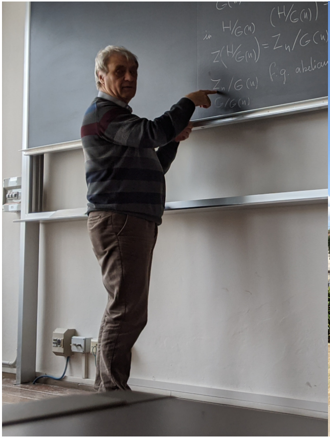
A few months later he was dead.




A
CARLO CASOLO
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21 FEBBRAIO 1958
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I will tell you some of the results.

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- A precise characterization of the set of natural numbers n for which every group of order n is integrable: these are the cubefree numbers n which do not have prime divisors p and q with $q \mid p - 1$. (This is similar to the condition for every group of order n to be cyclic, whose asymptotics were worked out by Paul Erdős.)

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- An abelian group of order n has an integral of order at most $n^{1+o(1)}$, but may fail to have an integral of order bounded by cn for any constant c.

Abelian groups

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Many, but not all, abelian groups have finite index in some integral.

Profinite groups

Profinite groups come with a natural topology. So we have to distinguish between the abstract derived group (the subgroup generated by commutators) and the topological derived group (its closure). We will say that a profinite group has a profinite integral if it is the topological derived group of a profinite group K.

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The example for the last assertion is the unrestricted Cartesian product of countably many copies of the dihedral group of order 8.

Varieties of groups

If $\mathfrak{V}$ is a variety of groups, then the set of all integrals of groups in $\mathfrak{V}$ is a variety; indeed it is the product variety $\mathfrak{V}\mathfrak{A}$, where $\mathfrak{A}$ is the variety of abelian groups. We call this the integral of $\mathfrak{V}$.

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We say that an identity $w = 1$ has gauge k if, whenever S is a generating set for a group G closed under inverses, and the identity $w = 1$ holds in the ball of radius k about the identity in the Cayley graph, then it holds in G.

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The varieties of abelian groups of exponent dividing m or nilpotent groups of class at most c have gauge 1, but the variety of metabelian groups has infinite gauge.

Inverse group theory

After some time exploring integrals of groups, we turn to a wide generalisation. Let $\mathcal{F}$ be a group-theoretic construction, so that for any group G there is a group $\mathcal{F}(G)$. (We do not assume any functorial properties of $\mathcal{F}$, merely isomorphism-invariance.) The inverse problem for $\mathcal{F}$ is: given a group G, is there a group H such that $G \cong \mathcal{F}(H)$?

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- For any finite group G, the Fitting subgroup of G is nilpotent; but every nilpotent finite group is the Fitting subgroup of a finite group (namely itself).

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Note that this proposition does *not* cover the Frattini subgroup.

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Recall that the Schur multiplier $M(G)$ of the finite group G is the (unique) largest abelian group Z for which there exists a group H with $Z \leq Z(H) \cap H'$ and $H/Z \cong G$. There are of course many other definitions.

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Proof.

A theorem of Schur says that

$$M(G \times H) = M(G) \times M(H) \times (G \otimes H).$$

Now $G \otimes H$ vanishes if G and H are perfect, so it is enough to realise arbitrary cyclic groups as Schur multipliers of perfect groups. Now C_n is the Schur multiplier of $\mathrm{PSL}(n, p)$ if $p \equiv 1 \pmod{n}$ with a few small exceptions; and Dirichlet's theorem guarantees infinitely many such primes p. □

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Using the fact that, if $a \leq b$, then $C_{p^a} \otimes C_{p^b} = C_{p^a}$, an easy induction using Schur's formula for direct products shows that, if G is the direct product of cyclic groups of orders $p^{a_1}, p^{a_2}, \ldots, p^{a_r}$, where the a_i are in nondecreasing order, then $M(G)$ is the direct product of $r-1$ copies of $C_{p^{a_1}}$, $r-2$ copies of $C_{p^{a_2}}, \ldots,$ and one copy of $C_{p^{a_{r-1}}}$.

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Using the fact that, if $a \leq b$, then $C_{p^a} \otimes C_{p^b} = C_{p^a}$, an easy induction using Schur's formula for direct products shows that, if G is the direct product of cyclic groups of orders $p^{a_1}, p^{a_2}, \ldots, p^{a_r}$, where the a_i are in nondecreasing order, then $M(G)$ is the direct product of $r-1$ copies of $C_{p^{a_1}}$, $r-2$ copies of $C_{p^{a_2}}, \ldots$, and one copy of $C_{p^{a_{r-1}}}$.
So, in particular, $C_p \times C_p$ is not the Schur multiplier of any finite abelian group. (However, we note that $C_2 \times C_2$ is the Schur multiplier of a finite simple group, for example $\mathrm{Sz}(8)$.)

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Another curious consequence of this classification is that a finite abelian group is isomorphic to its Schur multiplier if and only if it is the cube of a cyclic group.

Derangements

A derangement is a permutation with no fixed points. A century and a half ago, Camille Jordan showed that a finite transitive permutation group of degree greater than 1 must contain a derangement; later, Arjeh Cohen and I showed there must be many derangements (at least a fraction $1/n$ of the group elements, where n is the degree).

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One class of groups which do not are Frobenius groups, those in which the two-point stabiliser is trivial. Frobenius' Theorem shows that, in such a group, the identity and the derangements form a regular normal subgroup, so $G/D(G)$ is isomorphic to the Frobenius complement, the point stabiliser. Frobenius complements have a very restricted structure, which was worked out by Zassenhaus in the 1930s.

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With Rosemary Bailey, Michael Giudici and Gordon Royle, I looked at this question. Clearly every Frobenius complement occurs as a derangement quotient. Are there any others? We were able to find a few, for example the Klein group V_4 and the symmetric group S_3. (All other groups of order less than 8 are cyclic, and hence are Frobenius complements.) We found a few more too.

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A better approach came as a result of a talk I gave at the Ischia Group Theory conference last year. Carlo Scoppola was in the audience, and connected this question to …

The Frobenius–Wielandt theorem

Frobenius' Theorem has a purely group-theoretic statement. Let H be a nontrivial proper subgroup of the finite group G, and suppose that $H \cap H^g = 1$ for all $g \notin H$. Then the identity together with the elements in no conjugate of H is a normal subgroup, and H is a complement.

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In this form the statement was generalised by Wielandt, and Carlo knew this result well. Together we were able to link the two approaches, find new examples of derangement quotients, and put strong restrictions on the case where these have prime power order.

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The results can be found in

- R. A. Bailey, Peter J. Cameron, Norberto Gavioli and Carlo Maria Scoppola, The derangements subgroup in a finite permutation group and the Frobenius–Wielandt theorem, *Proceedings in Mathematics and Statistics*, in press

Cauchy numbers

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For example, 6 is a Cauchy number: a group with order divisible by 6 must contain one of the groups $\{C_6, S_3, A_4\}$.

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This can be found in the following paper:

- Peter J. Cameron, David Craven, Hamid Reza Dorbidi, Scott Harper and Benjamin Sambale, Minimal cover groups, *J. Algebra* **660** (2024), 345–372; doi: `10.1016/j.jalgebra.2024.06.038`

Other domains

Other kinds of structure can be constructed from groups; this gives rise to inverse group theory in other domains. I give two examples, one old, one new.

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A Latin square is the Cayley table of a group if and only if it satisfies the quadrangle condition.

The commuting graph

The commuting graph of a finite group G is the graph with vertex set G, in which x and y are joined if and only if $xy = yx$. This was introduced by Brauer and Fowler in 1955, in their seminal work on centralisers of involutions in finite simple groups.

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- V. Arvind, P. J. Cameron, X. Ma and N. Maslova, Aspects of the commuting graph, *J. Algebra*, in press; doi: `10.1016/j.jalgebra.2025.07.020`

… for your attention.