Semper Abstracta?

Peter J. Cameron
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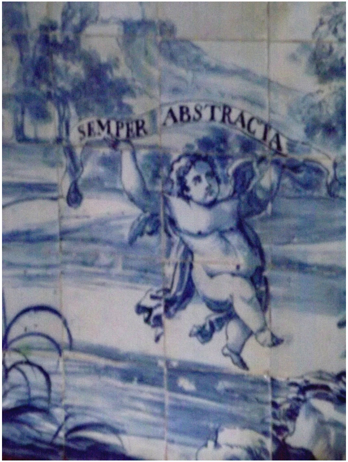

Theoretical and Computational Algebra
Évora, July 2025

João Araújo

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- masterminding this beautiful conference.

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In the interview for my postdoctoral fellowship in Oxford in 1971, I was asked about some discussions I had had with Don Taylor about a configuration of points in 23-dimensional Euclidean space (a subset of the Leech lattice). We thought that this was extremely concrete. But the committee (mostly non-mathematicians) were so impressed that somebody could think in 23 dimensions that I got the position.

Concrete or abstract?

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I am not so sure, however.

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Rings are fairly concrete objects, since there is a prototype or exemplar, the ring of integers, with which all our students are familiar. Once they have mastered an axiomatic system with the integers to guide them, we can make things more abstract, by stripping away axioms, to study groups, and then maybe semigroups.

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If you begin with groups, there is no natural exemplar.

Abstract or concrete? Road Closure

Abstract

A permutation group G on Ω has the Road Closure Property if no orbital graph for G can be disconnected by deleting the edges in a block of imprimitivity for G in its action on edges.

Concrete

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Mathematics becomes more concrete as it moves from mathematician to physicist, engineer, economist, etc. Also as it moves from general to particular. For example, “A group having exactly 59 subgroups” is surely more abstract than “The alternating group on 5 letters”, although they are exactly equivalent.

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A mathematician, at the start of a lecture, says “Let G be a group,” and a group is called into existence in the minds of the listeners, in a rather undefined form at first; but as the lecture proceeds, the group takes on more specific properties, and perhaps by the end of the lecture it has become a single group, or a narrowly specified class of groups.

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- the random graph and Zermelo–Fraenkel set theory,
- a problem about derangements,
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I will start each new section by showing you a picture of Évora, so if you have lost the thread you can come back in at that point.

The random graph and ZF set theory

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The countable random graph was a discovery of Erdős and Rényi. There is a graph R with the following remarkable property:

> *If you take a countable set of vertices, and for each pair of vertices, choose independently at random (for example, by tossing a coin) whether this pair are joined by an edge or not. Let X be the resulting random graph. Then X is almost surely isomorphic to R (that is, $\mathbb{P}(X \cong R) = 1$).*

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This already much more concrete than ZF set theory (although it is a non-constructive existence proof of R, a fairly abstract idea). But it can be made more concrete still. Independently of Erdős and Rényi, Rado constructed a graph R as on the next slide.

Rado's construction

Rado defined his graph as follows. Let $\mathbb{N}$ be the set of natural numbers (including 0). Rado's graph has vertex set $\mathbb{N}$. The rule for joining i and j is as follows. Suppose that $i < j$ (as we can always do). Now express j in base 2. If the ith digit is 1, we join i to j, otherwise not.

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Theorem

Rado's graph is isomorphic to R.

The connection

According to the downward Löwenheim–Skolem Theorem, if a set of first-order sentences in a countable language has a model, then it has a countable model. So, if ZF is consistent (as we all hope!), it has a countable model. This is despite the fact that a theorem of ZF is the existence of uncountable sets! This is the Skolem paradox, which I am not going to resolve here.

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So the abstract touches the very concrete (Rado's construction).

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In particular, if we take Rado's graph, and direct the edges from smaller to larger, we obtain a model of hereditarily finite set theory (obtained by replacing the Axiom of Infinity by its negation): all sets are finite, and all their members are finite, and so on all the way down.

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The sequel to this story is that I had a project student at St Andrews, Bea Adam-Day, who was fascinated by this story, and decided to do a project on what graphs are obtained if we replace the crucial Axiom of Foundation (forbidding infinite descending chains under membership) by the so-called Anti-Foundation Axiom. She did a good project, and then during her PhD at the University of Leeds, she and two fellow students essentially solved the problem completely.

Inverse group theory

This is an ongoing project with João Araújo, Francesco Matucci, and others.

Let F be an operation taking groups to groups (*not* necessarily a functor). The problem of inverse group theory is:

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Example Suppose that $F(G)$ is the derived group G' (the subgroup generated by commutators). A group G with $G' = H$ is called an integral of H.

Scandalously, we do not even know whether the computational problem of deciding whether a finite group is integrable is even decidable, let alone have a good estimate for its complexity!

Frobenius groups

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- *Let G be a finite transitive permutation group. Suppose that the stabiliser of any point is non-trivial, but the stabiliser of any two points is trivial. Then the identity and the derangements (fixed-point-free elements) in G form a regular normal subgroup, and the point stabilisers are complements.*

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- *Let G be a finite group with a subgroup H such that $H \neq 1$ but $H \cap H^g = 1$ for all $g \notin H$. Then H has a normal complement.*

The derangement subgroup

A few years ago, inspired by work of H. Zantema in number theory, Rosemary Bailey, Michael Giudici, Gordon Royle and I looked at the more general question. Let G be a finite transitive permutation group, and let $D(G)$ be the subgroup generated by the derangements in G. Which groups can arise as $G/D(G)$? (This fits well our paradigm of inverse group theory.)

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We found that the vast majority of transitive groups have $D(G) = G$. Of course, if G is a Frobenius group, then $D(G)$ is the Frobenius kernel, and the quotient is the Frobenius complement. The structure of Frobenius complements is known very precisely following work of Zassenhaus.

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We wondered at first whether every derangement quotient is isomorphic to a Frobenius complement, but found a few examples where this was not the case.

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The general problem of determining all derangement quotients is still open.

The ADE affair

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The classification of finite-dimensional simple Lie algebras over the complex numbers was achieved by Cartan and Killing in the early 20th century. It turns out to reduce to a very concrete problem.

Root systems

A Cartan subalgebra of a Lie algebra is a nilpotent self-normalising subalgebra. For simple Lie algebras over $\mathbb{C}$, this subalgebra is abelian (the Lie bracket is trivial on it), but has a natural Euclidean inner product; it also contains a finite set of vectors called a *root system*, the roots being eigenvectors of the adjoint transformation induced by elements outside.

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- the reflection in the hyperplane perpendicular to any one of its vectors maps S to itself.

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It is clear from the definition that the roots of fixed length in a root system form a root system in their own right. So as a first step we assume all roots have the same length, which we take to be $\sqrt{2}$. Then the inner product of two independent roots is in $\{0, \pm 1\}$, so they make angles 90°, 60° or 120°.

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Moreover, we can choose a basis such that all inner products are non-positive.
The Gram matrix of inner products of this set thus has the form $2I - A$, where A is a symmetric $0, 1$-matrix, hence the adjacency matrix of a graph whose greatest eigenvalue is less than 2.

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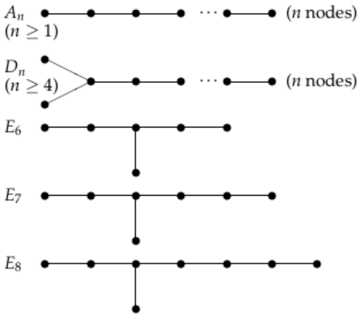

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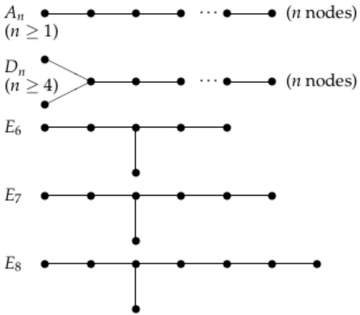

The other root systems (B_n, C_n, G_2 and F_4) can now be found by solving an easy puzzle involving putting together scaled copies of direct sums of these.

More

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My own first involvement with them was in the paper with Jean-Marie Goethals, Jaap Seidel, and Ernie Shult, where we settled a conjecture of Alan Hoffman by finding the graphs with least eigenvalue -2. This is not the dual of graphs with greatest eigenvalue 2 (there are many such graphs), but the ADE structures are used in the proof (we also gave a new proof of the classification).

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It is the first book in the series not to have a plain blue cover: the publisher and the LMS agreed to put the ADE diagrams on the cover.

Conclusion

The purpose of life is to prove and to conjecture.

Paul Erdős

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