Six views of discrete mathematics through the window of the Shrikhande graph

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International conference on discrete mathematics
CUSAT, June 2025

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- any two vertices, adjacent or not, have 2 common neighbours.

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Construction, uniqueness, automorphisms

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This gives us $1 + 6 + 9 = 16$ vertices, so we have all. The 9 vertices can be represented as v_{xy} where xy is an edge or long diagonal of H; and we only need to determine their adjacencies edges among these nine vertices.

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Hence, by the Orbit-Stabilizer Theorem, the automorphism group of the Shrikhande graph has order $16 \cdot 12 = 192$.

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The vertex set of our graph G is the group A. We join $v = (x, y, z)$ to $v' = (x', y', z')$ if $v' = v + s$, where s is in the following set:

$$S = \{(1, -1, 0), (-1, 1, 0), (0, 1, -1), (0, -1, 1), (-1, 0, 1), (1, 0, -1)\}.$$

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Since S is closed under taking inverses, the graph is undirected. (If $v' = v + s$, then $v = v' - s$.) Clearly it has valency 6. The fact that any two vertices have two common neighbours requires some checking.

We can see from this construction that the chromatic number is 4, as follows. Let us use e and o for an even (resp. odd) member of $\mathbb{Z}_4$. Then the vertex set of G can be divided into four sets as follows:

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If there were a colouring with three colours, there would have to be an independent set of size at least 6, and it is easy to see that no such set exists.
Also, the neighbourhood of a vertex is a hexagon, so there is no 4-clique. Thus, the graph is not weakly perfect.

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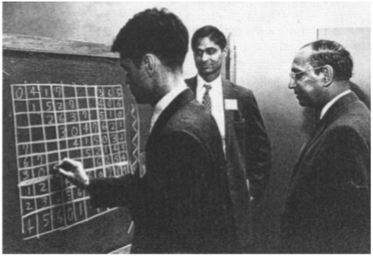

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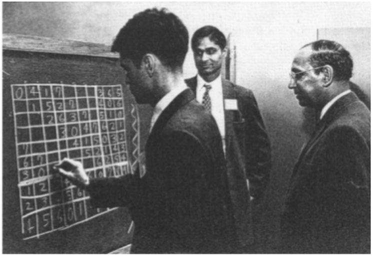

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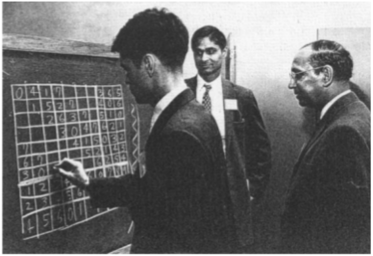

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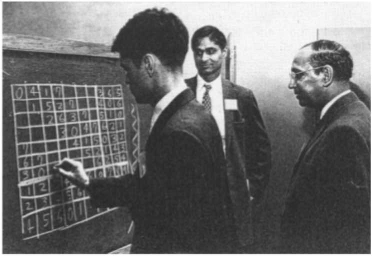

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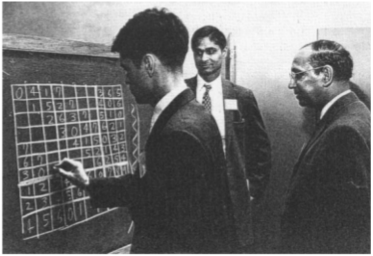

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A Latin square gives us a Latin square graph, whose vertices are the cells of the array, two vertices joined if they lie in the same row or the same column or contain the same symbol.

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There are two Latin squares of order 4, up to the obvious notion of isomorphism:

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The complements of their Latin square graphs are the Shrikhande graph and $L(K_{4,4})$.

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Euler had conjectured that orthogonal Latin squares exist if and only if n is not congruent to 2 (mod 4). This was refuted by Bose, Shrikhande and Parker who showed that they exist for all n except 2 and 6.

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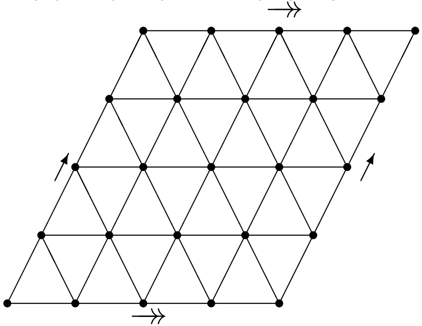
However, it can be embedded in the torus:



The arrows at the side show the identifications to be made.

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In the case of the Shrikhande graph, there are 192 flags, and 192 map automorphisms (for the faces are all the triangles in the graph, so are invariant under all graph automorphisms); so it is a regular map.

The Dyck graph

The icosahedron can be drawn as a regular map on the sphere. If we put a new vertex in the centre of each face, and join two new vertices if their faces meet on an edge, we obtain the dodecahedron, also as a regular map, which is dual to the icosahedron.

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We obtain a graph dual to the Shrikhande graph, called the Dyck graph, discovered in the 1880s. It has 32 vertices and 16 hexagonal faces; it is regular with degree 3 and girth 6. Its automorphism group is the same as that of the Shrikhande graph, of order 192.

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The proof came from an unexpected direction …

Generalized line graphs

The line graph $L(H)$ of a graph H is the graph whose vertex set is the edge set of H, two vertices joined if (as edges of H) they have a common vertex.

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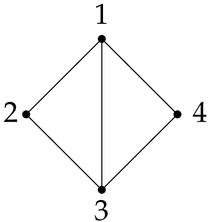
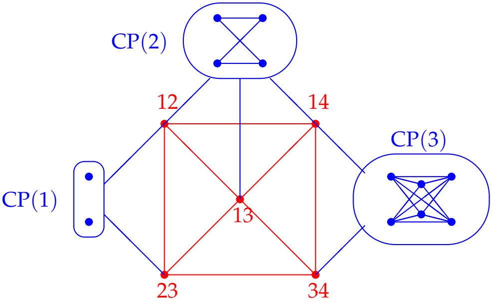
The next slide shows an example.

A generalized line graph

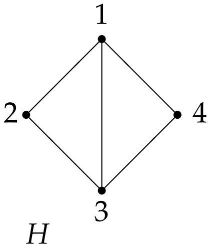


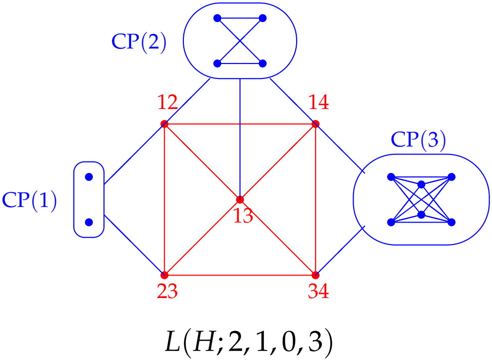
H



$L(H; 2, 1, 0, 3)$

A generalized line graph





The red part is the line graph $L(G)$; the blue shows the added cocktail party graphs.

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We close the system by adding the negatives of the vectors, and adding those vectors forming a star with existing vectors (i.e. if we have two vectors at an angle 60°, we add the vectors making angles 60° or 120° with both.

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The exceptions are not all classified, but all the regular graphs represented in E_8 have been determined. There are 187 of them which are not line graphs.

Later Hoffman proved that a connected graph with least eigenvalue greater than $-1-\sqrt{2}$ and sufficiently large valency is a generalized line graph. Very recently, Acharya and Jiang have improved our theorem a little bit by showing that all but finitely many connected graphs with least eigenvalue greater than $-2.01980\ldots$ are generalized line graphs.

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In the book on the Shrikhande graph, we include an explicit construction of this graph as a subset of the E_7 root system.

Seidel switching

Let $G = (V, E)$ be a graph, and $\{A, B\}$ a partition of V (we allow one of the parts to be empty). The result of Seidel switching of G with respect to the partition is obtained by interchanging edges and non-edges between A and B, leaving edges within either set unaltered.

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Several other combinatorial objects are equivalent to switching classes, including double covers of complete graphs, and sets of lines through the origin in Euclidean space such that any two make the same (supplementary) pair of angles.

The switching class of SG

The Shrikhande graph lies in a particularly interesting switching class, which also contains the line graph of $K_{4,4}$, the line graph of K_6 with an isolated vertex, and the Clebsch graph (a strongly regular graph with parameters $(16, 10, 6, 6)$).

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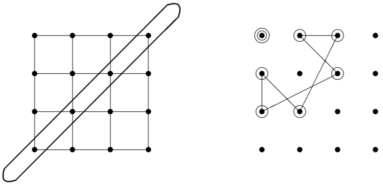
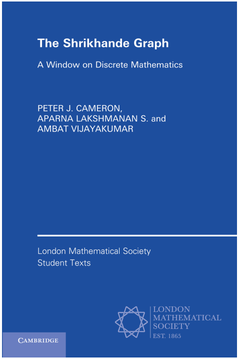
This gives us our final construction of SG: take the 4×4 square lattice graph, and switch with respect to a diagonal set:



The picture shows a switching set in the 4×4 square lattice, and a vertex neighbourhood in the switched graph (a 6-cycle).

This has been a whistle-stop tour through several topics in discrete mathematics. If you enjoyed it, you might like our book:



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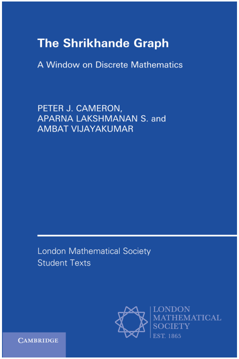

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