Permutation groups, partitions, and the Krasner–Kaloujnine theorem

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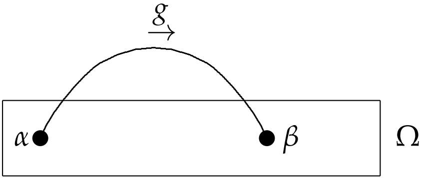

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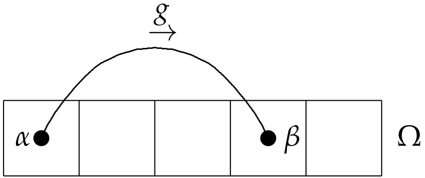

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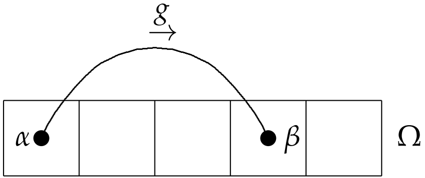

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If G is transitive and Π is invariant under G, then all parts of Π must have the same number of points: we say that Π is uniform.

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Construction

Let Π be a uniform partition of Ω. Let G_1 be a transitive permutation group whose degree is the size of a part of Π, and G_2 a transitive permutation group whose degree is the number of parts of Π. Now build a group G as follows: G is generated by

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- *permutations of Ω fixing every part of Π and acting on each part as an element of G_1;*
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This group is the wreath product of G_1 and G_2, denoted by $G_1 \wr G_2$.

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Theorem

Let G be a transitive permutation group on Ω, having an invariant partition of Π. Let G_1 be the group induced on a part of Π by its setwise stabiliser, and G_2 the group induced on the set of parts by G. Then G is naturally embedded as a subgroup of $G_1 \wr G_2$.

The lattice of partitions

Let Π_1 and Π_2 be partitions. We say that Π_1 refines Π_2 if every part of Π_1 is contained in a part of Π_2 (written $\Pi_1 \preceq \Pi_2$).

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The partitions invariant under any group form a lattice.

Commmuting partitions

Another structure on partitions comes from noting that each partition Π is defined by an equivalence relation R; we will say that partitions commute if the corresponding equivalence relations do, in the sense that $R_1 \circ R_2 = R_2 \circ R_1$, where

$$R_1 \circ R_2 = \{(\alpha, \gamma) : (\exists \beta)(\alpha, \beta) \in R_1, (\beta, \gamma) \in R_2\}.$$

This notion is important in statistics since it means that the factors corresponding to the two partitions are statistically orthogonal, which makes the analysis much easier.

Distributive lattices

We will be interested in lattices that satisfy the distributive laws, the following (equivalent) statements:

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The structure theorem for these asserts that any finite distributive lattice Λ is embeddable in the Boolean lattice of subsets of a set; more exactly, there is a poset (partially ordered set) $(\mathcal{P}, \sqsubseteq)$ such that Λ is isomorphic to the lattice of down-sets (sets closed downwards) of $\mathcal{P}$.

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In the other direction, $\mathcal{P}$ is the subset of join-indecomposable elements of Λ (those which cannot be written as the join of smaller elements).

Distributive lattices and poset block structures

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Let us suppose that we have a group G_p acting transitively on Ω_p for each $p \in P$. Then there is a construction called the generalized wreath product, first described by Charles Holland, which produces a permutation group G on the Cartesian product Ω of the sets Ω_p over $p \in P$. Unfortunately, it is too complicated to describe here.

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Note that, if $\mathcal{P}$ is a 2-element chain, then the g.w.p is just the wreath product defined earlier, while if $\mathcal{P}$ is a 2-element antichain then the g.w.p is the direct product.

Corresponding to each element $\lambda \in \Lambda$, the lattice of down-sets in $\mathcal{P}$, we have a partition Π_λ of Ω; and these partitions commute pairwise.

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A structure like this (a distributive lattice of pairwise commuting uniform partitions) is known to statisticians as a poset block structure.

The main theorem

Our main theorem generalizes the Krasner–Kaloujnine theorem.

Theorem

Suppose that G is a permutation group on Ω, which has a set of invariant partitions forming a poset block structure; let $\mathcal{P}$ be the corresponding poset.
Then it is possible to extract a group G_p for every $p \in \mathcal{P}$ such that G is naturally embedded in the generalized wreath product of the groups G_p over $p \in \mathcal{P}$.

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Two closing remarks:

1. There is some subtlety in the construction of the groups G_p; the straightforward construction in the Krasner–Kaloujnine theorem does not work.
2. We cannot weaken the hypotheses of the theorem. Even deleting "distributive" in the definition of a poset block structure (giving a more general class of objects called orthogonal block structures in statistics) does not work.