Two new digraphs defined on groups

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Graphs on groups

There has been a lot of work recently on graphs defined on groups. This began with Cayley in the 19th century. To obtain a Cayley graph, we choose a subset S of G which is inverse-closed and doesn't contain the identity, and join x to y if $xy^{-1} \in S$. This group admits an action of G by right multiplication.

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My topic is a bit different. I am considering graphs where the adjacency is defined purely in terms of group-theoretic properties of G. These graphs admit the automorphism group of G as automorphisms; in particular, G acts by conjugation.

The commuting graph and others

The first example was the commuting graph, where x and y are joined if $xy = yx$. Brauer and Fowler used this to prove an important theorem which was perhaps the first step to the Classification of Finite Simple Groups: they showed that there are only finitely many finite simple groups with a given involution centraliser.

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I have talked about some of this here before.

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The second was with three north Indian mathematicians, Rishabh Chakraborty, Rajat Kanti Nath and Deiborlang Nongsiang, and concerns the Engel digraph. This is in a way more specialised, but digs more deeply into the group theory.
The study of the undirected version (the Engel graph) is not entirely new. It was first considered by Alireza Abdollahi in Iran, and then by Andrea Lucchini and some of his coauthors in Italy. However, our results on the directed version seem to be new.

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What is the analogous thing for a monoid action? We lose the inverse axiom, and so we lose the symmetric law.

So a partial preorder on a set X is a reflexive and transitive relation on X. I will write $x \to y$ for a partial preorder, to emphasize that it is a special kind of digraph (with a loop at every vertex).

Properties of partial preorders

Exercise

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Let $\to$ be a partial preorder on X. Define a relation $\equiv$ by $x \equiv y$ if $x \to y$ and $y \to x$. Show that $\equiv$ is an equivalence relation, and the equivalence classes are partially ordered by $\to$.

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Partial preorders are sometimes called *preferential arrangements*; you are asked to rank, say, politicians, but there are some subsets which you are unable to order. Thus the equivalence classes of $\equiv$ are called indifference classes.

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For, given a partial preorder, there is a partial order with the same comparability graph (simply put a total order on each indifference class). A theorem of Mirsky asserts that comparability graphs of partial orders are perfect.

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If A has the property that $\langle x \rangle$ is the set of positive integer powers of x (as semigroups and finite groups do), then we can simply say $x \to y$ if $y = x^m$ for some positive integer m; so the partial preorder corresponds to the action of the multiplicative monoid of positive integers on A.

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The preorder with loops deleted is the directed power graph of A, and its comparability graph is the (undirected) power graph.

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Let M be the endomorphism monoid of A. The endomorphism digraph of A is the orbit partial preorder of M on A (in other words, $x \to y$ if there is an endomorphism in M which maps x to y), thought of as a digraph by deleting the loops. The endomorphism graph is the comparability graph (that is, we ignore the directions on edges).

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- *If A is an abelian group, then the power graph is a spanning subgraph of the endomorphism graph.*
- *If A is a cyclic group, then the power graph is equal to the endomorphism graph.*

An exercise

The second part of the above proposition holds because, in an abelian group, the power maps $f_m : x \mapsto x^m$ are endomorphisms. This is not true for general groups. You might enjoy the following exercise, if you have not seen it before:

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In particular, since f_0 and f_1 are (trivially) endomorphisms, we recover the standard results that if either f_2 or f_{-1} are endomorphisms then G is abelian.

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For example, let A_1 and A_2 be the two groups of order p^2 where p is prime. For the elementary abelian group, the automorphism group acts transitively on the non-identity elements; so the endomorphism digraph is the complete digraph on the non-identity elements together with the identity as a sink, and the endomorphism graph is complete.

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On the other hand, for the cyclic group, an element of order p^2 can be mapped to any element; an element of order p can be mapped to any element of order 1 or p. So again the endomorphism graph is complete, but the endomorphism digraph is not the same as for the elementary abelian group.

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- *For which groups does the (directed or undirected) endomorphism graph determine the group up to isomorphism?*
- *For which groups do the orbits of the automorphism group coincide with the indifference classes of the endomorphism preorder?*
- *Investigate properties of the endomorphism graph such as clique number and independence number.*

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In this case, the undirected graph determines the directed graph up to isomorphism.
Among many results known about this graph, I mention just one. Let $f(n)$ be the clique number of the power graph of C_n.

Theorem

$$\phi(n) \leq f(n) \leq c\phi(n),$$

where ϕ is Euler's function, and $c = 2.6481017597\ldots$

Commutators, nilpotent and Engel groups

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A group is k-Engel if it satisfies $[x, {}_k y] = 1$ for all x and y, where $[x, {}_k y] = [x, y, \ldots, y]$ with k occurrences of y. It is Engel if it is k-Engel for some k.

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Theorem
A finite Engel group is nilpotent.

The Engel digraph and the nilpotency graph

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For the forward direction, we note that if a group is not nilpotent, then it contains a minimal non-nilpotent group as a subgroup; Schmidt classified these groups, and showed that each can be generated by two elements.

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This requires a little more knowledge of Schmidt's minimal non-nilpotent groups. The *Fitting subgroup* $F(G)$ of a finite group G is the (unique) maximal normal nilpotent subgroup. If G is not nilpotent, then $F(G) \neq G$, and using Schmidt's result, we can find a directed arc from a vertex in $F(G)$ to a vertex outside $F(G)$.

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The first and second parts are due to Baer; the third to Abdollahi.

Some questions

An example of a group whose directed Engel graph is not a partial preorder is the symmetric group S_4. We have $(1,2)(3,4) \to (1,2,3) \to (1,2)$, and also $(1,2) \to (1,2)(3,4)$, but $(1,2) \not\to (1,2,3)$ and $(1,2,3) \not\to (1,2)(3,4)$.

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- *Does the theorem characterising finite nilpotent groups hold without the assumption of solubility?*
- *Which groups are characterised up to isomorphism by their Engel digraphs?*
- *Which groups have the property that every single arc in the Engel digraph has its initial vertex in the Fitting subgroup?*

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Conversely, if x is in the hypercentre, then $\langle x, y \rangle$ is nilpotent for all $y \in G$, by induction on the length of the lower central series (using the fact that, if Z is a subgroup in the centre of G, and G/Z is nilpotent, then G is nilpotent).

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