Graphs on groups and algebras

Peter J. Cameron
University of St Andrews

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Groups and graphs

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The first example was the commuting graph of G; its vertex set is G, or possibly $G \setminus Z(G)$, and two vertices are joined if they commute. This was used by Brauer and Fowler in their celebrated paper on centralizers of involutions in simple groups of even order, arguably the first step towards the Classification of Finite Simple Groups.

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3. They were interested in questions of connectedness and diameter, so they removed the centre (just the identity if the group is simple). For other questions such as clique and chromatic number and induced subgraphs, it makes little difference.
4. What we regard as their main result, that there are only finitely many finite simple groups with a given involution centralizer, is not even stated formally as a theorem in their paper.

Graphs on groups

Apart from a couple of papers on the power graph in 2010-11, my involvement began during the Covid pandemic, in 2021. First, I wrote a long survey paper and put it on the arXiv (it was later published in the *International Journal of Group Theory*); and then Ambat Vijayakumar saw the arXiv paper and asked me to lead an on-line research discussion on it. The result is that many of my coauthors are Indian, and that country probably leads the world now in this area.

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There is no time to give a detailed survey of this field. Many of the results are about properties of particular graphs on particular groups; I would rather talk about general principles. I see four types of fruitful interaction, of which I will briefly discuss examples.

1. We may follow Brauer and Fowler and prove new results about groups using graphs as a tool. One example is a strengthening of Landau's old result that there are only finitely many finite groups with a given number of conjugacy classes.

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2. Graphs can be used to define interesting classes of groups, either by restricting the graphs to lie in an interesting graph class, or by asking for two different graphs on the group to coincide.
3. We may find beautiful and perhaps useful examples of graphs.
4. There are many interesting computational problems along the way, such as the complexity of recognising a certain type of graph on groups, or determining groups which give rise to a particular graph.

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They proved a number of results about it, and I hope that it will be developed and will take its place as a concept in mainstream finite group theory.

2. Classes of groups

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These groups can also be described as those whose Gruenberg–Kegel graph has no edges.

3. Beautiful graphs

Sometimes we have to strip away rubbish from a graph to reveal the beauty lying within. Here, this is done by twin reduction. Two vertices are twins if they have the same neighbours apart possibly from one another. Twin reduction means finding and identifying pairs of twins until none remain; the result is independent of the reduction process.

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For example, take G to be the Mathieu group M_{11}; the graph is the difference graph, whose edges are those of the enhanced power graph not lying in the power graph. The result of twin reduction is a semiregular bipartite graph on $165 + 220$ vertices, with valencies 4 and 3 in the two parts; it is connected with diameter 10 and girth 10 (this is rather large).

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This is part of a paper (with other authors Natalia Maslova and Xuanlong Ma) dedicated to Richard Parker.

Digraphs

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- The Engel digraph, with an arc from x to y if $[y,_k x] = 1$ for some k, where $[y,_k x]$ is the iterated commutator $[y, x, \ldots, x]$ (with k entries x.

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In each case, we can form an undirected graph if we ignore directions on arcs and replace any double edges that result by single edges.

Some results about digraphs

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This was proved by Samir Zahirović and colleagues.

Simplicial complexes

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The strong simplicial complex is contained in the simplicial complex. The groups for which they coincide were determined by Andrea Lucchini and Mima Stanojkovski.

Other algebras

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However, there are many graphs which depend only on the notion of subalgebra and/or endomorphism, and can be extended to any algebra. These include the generating graph, independence and strong independence complexes, and with a little rephrasing, the power graph and enhancend power graph.

Surprisingly it turns out that many of the results extend …

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For the enhanced power graph, we have a choice: we could either join x and y if $\langle x, y\rangle$ is 1-generated, or if there exists z such that $x, y \in \langle z\rangle$. These are equivalent for groups, but Misha Volkov pointed out to me that they are not equivalent for semigroups. The second definition is more useful, so I adopt it.

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(The second part is due to Bubbolini, Fumagalli and Praeger for groups.)

References

- Peter J. Cameron, Graphs defined on groups, *Internat. J. Group Theory* **11** (2022), 43–124; doi: 10.22108/ijgt.2021.127679.1681
- Peter J. Cameron, Simplicial complexes defined on groups, *Open Mathematics* **23** (2025), 20250171; doi: 10.1515/math-2025-0171
- Peter J. Cameron, Graphs on algebras, *Adv. Group Theory Appl.*, in press; arXiv 2602.00712
- V. Arvind, P. J. Cameron, X. Ma and N. Maslova, Aspects of the commuting graph, *J. Algebra* **703** (2026), 4–64; doi: 10.1016/j.jalgebra.2025.07.020

References

- Peter J. Cameron, Graphs defined on groups, *Internat. J. Group Theory* **11** (2022), 43–124; doi: `10.22108/ijgt.2021.127679.1681`
- Peter J. Cameron, Simplicial complexes defined on groups, *Open Mathematics* **23** (2025), 20250171; doi: `10.1515/math-2025-0171`
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